HIGH ORDER A-STABLE BLOCK ETR_2s AND THEIR APPLICATION TO SYSTEM OF FIRST ORDER ODES

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ABSTRACT

An eighth order block Extended Trapezoidal Rule of Second kind (ETR_2s) is presented for the numerical integration of stiff system of first-order ordinary differential equations. In the derivation process, we adopt the power series approach which leads to a system of equations that would be solved simultaneously in block form to generate approximate solution for the differential equations. The stability properties of the method are also presented. Some test, reported to emphasize pros and cons of the method.

Keywords: Extended Trapezoidal Rule of Second kind, Stiff System, Power Series, A(α)-Stability Scientific Computation, A-Stable Method, Ordinary Differential Equations, and Collocation Method.

INTRODUCTION

Numerical solution for ordinary differential equations (ODE's) have great importance in scientific computation, as they were widely used to model in representing the real world problems. The commonly used methods to solve ODE's are categorized as one-step methods and multistep methods, which Runge-Kutta methods represent the former group and Adams-Bashforth/Moulton represent the latter group [9].

Over the years, several scholars have proposed the collocation method as ways of generating numerical solutions to ODE's. The collocation method is dated as far back as in the 1950's in the research conducted by Lanczos [12] and Brunner [3].

Scholars such as Onumanyi *et al*. [15], Fatunla [8], introduced other variants of the collocation method leading to the continuous collocation approach. The advantages of the continuous collocation method over the discrete ones include the following:

(i) Better global error can be estimated

(ii) Approximations can be obtained at all interior points.

Thus, the introduction of the continuous collocation method is to bridge the gap between the discrete collocation and the conventional multistep methods [14].

We seek to propose a new numerical integrator for the solution of first-order ordinary differential equation of the form:

$$y' = f(x, y(x)),\ y(x_0) = y_0,\ a \le x \le b \tag{1}$$

with initial conditions

where,f is continuous within the interval of integration, we assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1).

The general k-step method given by Lambert [11] is written in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j},\ \alpha_k \ne 0 \tag{2}$$

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where, α_j and β_j are coefficients of the method to be uniquely determined, h a constant step size and k the step number.

We propose in this study a basis function of the form:

$$y(x) = \sum_{j=0}^{m} \varphi_j (x - x_k)^j \tag{3}$$

Equation (3) can now be used to generate the high order A-stable block ETR$_2$s method.

DERIVATION PROCEDURE

Consider the polynomial function:

$$y(x) = \sum_{j=0}^{m} \varphi_j (x - x_k)^j \equiv y(x), x_k \le x \le x_{k+p} \tag{4}$$

Over each of the sub-interval (x_k, x_{k+p}) of (a,b) where, m is appropriately chosen. This shall be used as basis function to derive the LMM in continuous form.

The technique which is being employed is using the trial or basis function

$$Y(x) = \sum_{j=0}^{n+1} \varphi_j (x - x_k)^j \equiv y(x), x_k \le x \le x_{k+p} \tag{5}$$

This satisfies the unperturbed ODE:

$$\left.\begin{aligned} Y'(x) &= f(x, y(x)), x_k \le x \le x_{k+p} \\ Y(x_k) &= Y_k \end{aligned}\right\} \tag{6}$$

Collocating equation (6) at $(n-5)$ points $x_{k+j}, j = 3,4,\ldots,n$ and interpolating the trial polynomial (5) at $x_{k+j}, j = 0,1,2,\ldots,6$ to give the require $(n+2)$ equations for the unique determination of φ_j.

Doing this, we write

$$\left.\begin{aligned} f(x_{k+j}) &= f_{k+j}, j = 3,4,\ldots \\ Y(x_{k+j}) &= Y_{k+j}, j = 0,1,2,\ldots,6 \end{aligned}\right\} \tag{7}$$

To derive the eighth order block ETR$_2$s method, we set, $n = 7$ in the equation (5), so that

$$Y(x) = \varphi_0 + \varphi_1(x - x_k) + \varphi_2(x - x_k)^2 + \ldots + \varphi_8(x - x_k)^8 \tag{8}$$

From equation (7), we have

$$\left.\begin{aligned} Y'(x_{k+3}) &= f_{k+3} \\ Y'(x_{k+4}) &= f_{k+4} \\ Y(x_k) &= Y_k \\ Y(x_{k+1}) &= Y_{k+1} \\ &. \\ &. \\ &. \\ Y(x_{k+6}) &= Y_{k+6} \end{aligned}\right\} \tag{9}$$

Using equation 8 in 9, we obtain the following equations

$$Y(x_k) = \varphi_0 = Y_k$$

$$Y(x_{k+1}) = \varphi_0 + \varphi_1(\mu) + \varphi_2(\mu)^2 + \varphi_3(\mu)^3 + \ldots + \varphi_8(\mu)^8 = Y_{k+1}$$

$$Y(x_{k+2}) = \varphi_0 + 2\varphi_1(\mu) + 4\varphi_2(\mu)^2 + 8\varphi_3(\mu)^3 + \ldots + 256\varphi_8(\mu)^8 = Y_{k+2}$$

$$Y(x_{k+3}) = \varphi_0 + 3\varphi_1(\mu) + 9\varphi_2(\mu)^2 + 27\varphi_3(\mu)^3 + \ldots + 6561\varphi_8(\mu)^8 = Y_{k+3}$$

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$$Y(x_{k+6}) = \varphi_0 + 6\varphi_1(\mu) + 36\varphi_2(\mu)^2 + 216\varphi_3(\mu)^3 + \ldots + 1679616\varphi_8(\mu)^8 = Y_{k+6}$$

and

$$Y'(x_{k+3}) = \varphi_1 + 6\varphi_2(\mu) + 27\varphi_3(\mu)^2 + 108\varphi_4(\mu)^3 + \ldots + 174966\varphi_8(\mu)^7 = f_{k+3}$$

$$Y'(x_{k+4}) = \varphi_1 + 8\varphi_2(\mu) + 48\varphi_3(\mu)^2 + 256\varphi_4(\mu)^3 + \ldots + 131072\varphi_8(\mu)^7 = f_{k+4}$$

where,

$$\mu = x - x_k$$

Representing this in matrix form yield

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mu & \mu^2 & \mu^3 & \mu^4 & \mu^5 & \mu^6 & \mu^7 & \mu^8 \\ 1 & 2\mu & 4\mu^2 & 8\mu^3 & 16\mu^4 & 32\mu^5 & 64\mu^6 & 128\mu^7 & 256\mu^8 \\ 1 & 3\mu & 9\mu^2 & 27\mu^3 & 81\mu^4 & 243\mu^5 & 729\mu^6 & 2187\mu^7 & 6561\mu^8 \\ 1 & 4\mu & 16\mu^2 & 64\mu^3 & 256\mu^4 & 1024\mu^5 & 4096\mu^6 & 16384\mu^7 & 65536\mu^8 \\ 1 & 5\mu & 25\mu^2 & 125\mu^3 & 625\mu^4 & 3125\mu^5 & 15625\mu^6 & 78125\mu^7 & 390625\mu^8 \\ 1 & 6\mu & 36\mu^2 & 216\mu^3 & 1296\mu^4 & 7776\mu^5 & 46656\mu^6 & 279936\mu^7 & 1679616\mu^8 \\ 0 & 1 & 6\mu & 27\mu^2 & 108\mu^3 & 405\mu^4 & 1458\mu^5 & 5103\mu^6 & 174966\mu^7 \\ 0 & 1 & 8\mu & 48\mu^2 & 256\mu^3 & 1280\mu^4 & 6144\mu^5 & 28672\mu^6 & 131072\mu^7 \end{bmatrix}$$

$$\begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_7 & \varphi_8 \end{bmatrix}^T$$

$$= \begin{bmatrix} Y_k & Y_{k+1} & Y_{k+2} & Y_{k+3} & Y_{k+4} & Y_{k+5} & Y_{k+6} & f_{k+3} & f_{k+4} \end{bmatrix}^T$$

Solving for $\varphi_i, i = 1,2,\ldots,8$ we obtained

$$\varphi_0 = Y_k$$

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$$\varphi_1 = \frac{12}{h}Y_{k+1} - \frac{45}{h}Y_{k+2} - \frac{160}{3h}Y_{k+3} + \frac{165}{2h}Y_{k+4} + \frac{36}{5h}Y_{k+5} - \frac{1}{3h}Y_{k+6}$$
$$-80f_{k+3} - 45f_{k+4} - \frac{91}{30h}Y_k$$

$$\varphi_2 = \frac{114}{h^2}Y_{k+2} - \frac{122}{5h^2}Y_{k+1} + \frac{1376}{9h^2}Y_{k+3} - \frac{3629}{16h^2}Y_{k+4} - \frac{102}{5h^2}Y_{k+5} + \frac{43}{45h^2}Y_{k+6}$$
$$+\frac{216}{h}f_{k+3} + \frac{501}{4h}f_{k+4} + \frac{2713}{720h^2}Y_k$$

$$\varphi_3 = \frac{1249}{60h^3}Y_{k+1} - \frac{1801}{16h^3}Y_{k+2} - \frac{174}{h^3}Y_{k+3} + \frac{739}{3h^3}Y_{k+4} + \frac{461}{20h^3}Y_{k+5} - \frac{2369}{2160h^3}Y_{k+6}$$
$$-\frac{2065}{9h^2}f_{k+3} - \frac{553}{4h^2}f_{k+4} - \frac{343}{135h^3}Y_k$$

$$\varphi_4 = \frac{5573}{96h^4}Y_{k+2} - \frac{1741}{180h^4}Y_{k+1} + \frac{103}{h^4}Y_{k+3} - \frac{80287}{576h^4}Y_{k+4} - \frac{821}{60h^4}Y_{k+5} + \frac{2869}{4320h^4}Y_{k+6}$$
$$+\frac{1141}{9h^3}f_{k+3} + \frac{3829}{48h^3}f_{k+4} + \frac{8869}{8640h^4}Y_k$$

$$\varphi_5 = \frac{127}{48h^5}Y_{k+1} - \frac{103}{6h^5}Y_{k+2} - \frac{69}{2h^5}Y_{k+3} + \frac{2153}{48h^5}Y_{k+4} + \frac{1117}{240h^5}Y_{k+5} - \frac{25}{108h^5}Y_{k+6}$$
$$-\frac{1435}{36h^4}f_{k+3} - \frac{105}{4h^4}f_{k+4} - \frac{553}{2160h^5}Y_k$$

$$\varphi_6 = \frac{47}{16h^6}Y_{k+2} - \frac{307}{720h^6}Y_{k+1} + \frac{79}{12h^6}Y_{k+3} - \frac{2381}{288h^6}Y_{k+4} - \frac{73}{80h^6}Y_{k+5} + \frac{101}{2160h^6}Y_{k+6}$$
$$+\frac{259}{36h^5}f_{k+3} + \frac{119}{24h^5}f_{k+4} + \frac{167}{4320h^6}Y_k$$

$$\varphi_7 = \frac{3}{80h^7}Y_{k+1} - \frac{13}{48h^7}Y_{k+2} - \frac{2}{3h^7}Y_{k+3} + \frac{13}{16h^7}Y_{k+4} + \frac{23}{240h^7}Y_{k+5} - \frac{11}{2160h^7}Y_{k+6}$$
$$-\frac{25}{36h^6}f_{k+3} - \frac{1}{2h^6}f_{k+4} - \frac{7}{2160h^7}Y_k$$

$$\varphi_8 = \frac{1}{96h^8}Y_{k+2} - \frac{1}{720h^8}Y_{k+1} + \frac{1}{36h^8}Y_{k+3} - \frac{19}{576h^8}Y_{k+4} - \frac{1}{240h^8}Y_{k+5} + \frac{1}{4320h^8}Y_{k+6}$$
$$+\frac{1}{36h^7}f_{k+3} + \frac{1}{48h^7}f_{k+4} + \frac{1}{8640h^8}Y_k$$

Substituting in equation (8), we get

$$\left.\begin{aligned}
Y(x) = Y_k &+ \left(\frac{12}{h}Y_{k+1} - \frac{45}{h}Y_{k+2} - \frac{160}{3h}Y_{k+3} + \frac{165}{2h}Y_{k+4} + \frac{36}{5h}Y_{k+5} - \frac{1}{3h}Y_{k+6} - 80f_{k+3} - 45f_{k+4} - \frac{91}{30h}Y_k\right)\mu \\
&+ \left(\frac{114}{h^2}Y_{k+2} - \frac{122}{5h^2}Y_{k+1} + \frac{1376}{9h^2}Y_{k+3} - \frac{3629}{16h^2}Y_{k+4} - \frac{102}{5h^2}Y_{k+5} + \frac{43}{45h^2}Y_{k+6} + \frac{216}{h}f_{k+3} + \frac{501}{4h}f_{k+4} + \frac{2713}{720h^2}Y_k\right)\mu^2 \\
&+ \left(\frac{1249}{60h^3}Y_{k+1} - \frac{1801}{16h^3}Y_{k+2} - \frac{174}{h^3}Y_{k+3} + \frac{739}{3h^3}Y_{k+4} + \frac{461}{20h^3}Y_{k+5} - \frac{2369}{2160h^3}Y_{k+6} - \frac{2065}{9h^2}f_{k+3} - \frac{553}{4h^2}f_{k+4} - \frac{343}{135h^3}Y_k\right)\mu^3 \\
&+ \left(\frac{5573}{96h^4}Y_{k+2} - \frac{1741}{180h^4}Y_{k+1} + \frac{103}{h^4}Y_{k+3} - \frac{80287}{576h^4}Y_{k+4} - \frac{821}{60h^4}Y_{k+5} + \frac{2869}{4320h^4}Y_{k+6} + \frac{1141}{9h^3}f_{k+3} + \frac{3829}{48h^3}f_{k+4} + \frac{8869}{8640h^4}Y_k\right)\mu^4 \\
&+ \left(\frac{127}{48h^5}Y_{k+1} - \frac{103}{6h^5}Y_{k+2} - \frac{69}{2h^5}Y_{k+3} + \frac{2153}{48h^5}Y_{k+4} + \frac{1117}{240h^5}Y_{k+5} - \frac{25}{108h^5}Y_{k+6} - \frac{1435}{36h^4}f_{k+3} - \frac{105}{4h^4}f_{k+4} - \frac{553}{2160h^5}Y_k\right)\mu^5 \\
&+ \left(\frac{47}{16h^6}Y_{k+2} - \frac{307}{720h^6}Y_{k+1} + \frac{79}{12h^6}Y_{k+3} - \frac{2381}{288h^6}Y_{k+4} - \frac{73}{80h^6}Y_{k+5} + \frac{101}{2160h^6}Y_{k+6} + \frac{259}{36h^5}f_{k+3} + \frac{119}{24h^5}f_{k+4} + \frac{167}{4320h^6}Y_k\right)\mu^6 \\
&+ \left(\frac{3}{80h^7}Y_{k+1} - \frac{13}{48h^7}Y_{k+2} - \frac{2}{3h^7}Y_{k+3} + \frac{13}{16h^7}Y_{k+4} + \frac{23}{240h^7}Y_{k+5} - \frac{11}{2160h^7}Y_{k+6} - \frac{25}{36h^6}f_{k+3} - \frac{1}{2h^6}f_{k+4} - \frac{7}{2160h^7}Y_k\right)\mu^7 \\
&+ \left(\frac{1}{96h^8}Y_{k+2} - \frac{1}{720h^8}Y_{k+1} + \frac{1}{36h^8}Y_{k+3} - \frac{19}{576h^8}Y_{k+4} - \frac{1}{240h^8}Y_{k+5} + \frac{1}{4320h^8}Y_{k+6} + \frac{1}{36h^7}f_{k+3} + \frac{1}{48h^7}f_{k+4} + \frac{1}{8640h^8}Y_k\right)\mu^8
\end{aligned}\right\} \quad 10)$$

On evaluation of (10) at some end and interior points, we obtained the following equations whose coefficients are presented in table 1 and 2.

Table 1. Coefficients of $\hat{\alpha}_i, i = 0,1,2,\ldots,7$

Eqn.	η	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\alpha}_6$	$\hat{\alpha}_7$
1	840	$-1/\eta$	$14/\eta$	$-126/\eta$	$-525/\eta$	$525/\eta$	$126/\eta$	$-14/\eta$	$1/\eta$
2	72660	$-91/\eta$	$1264/\eta$	$-11250/\eta$	$-45300/\eta$	$46175/\eta$	$10116/\eta$	$-914/\eta$	
3	5040	$-5/\eta$	$72/\eta$	$-675/\eta$	$-3200/\eta$	$2925/\eta$	$1080/\eta$	$-197/\eta$	
4	1680	$-1/\eta$	$15/\eta$	$-150/\eta$	$-900/\eta$	$525/\eta$	$501/\eta$	$10/\eta$	
5	840	$1/\eta$	$-24/\eta$	$-375/\eta$		$375/\eta$	$24/\eta$	$-1/\eta$	
6	840	$5/\eta$	$127/\eta$	$-450/\eta$	$-300/\eta$	$575/\eta$	$45/\eta$	$-2/\eta$	
7	3780	$-91/\eta$	$360/\eta$	$-1350/\eta$	$-1600/\eta$	$2475/\eta$	$216/\eta$	$-10/\eta$	

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Table 2. Coefficients of $\hat{\beta}_{i,} i = 0,1,2,\ldots,7$

Eqn.	ω	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$	$\hat{\beta}_7$
1	2				$1/\omega$	$1/\omega$			
2	2422				$1229/\omega$			$1194/\omega$	
3	84	$-1/\omega$			$40/\omega$			$45/\omega$	
4	28	$-1/\omega$			$10/\omega$	$15/\omega$	$3/\omega$		
5	14			$3/\omega$	$8/\omega$	$3/\omega$			
6	14		$1/\omega$		$10/\omega$	$5/\omega$			
7	126	$1/\omega$			$80/\omega$	$45/\omega$			

STABILITY ANALYSIS OF THE HIGH ORDER A-STABLE BLOCK ETR$_2$S

In order to ascertain the accuracy and suitability of our method, we shall investigate its basic properties such as order of accuracy, error-constants, consistency, zero-stability and convergence.

Order of Accuracy and Error Constants

The local truncation error is computed using the equation

$$T_{n+k} = c_0 y_n + c_1 h y'_n + c_2 h^2 y''_n + c_3 h^3 y'''_n + \ldots + c_p h^p y_n^{(p)} + c_{p+1} h^{p+1} y_n^{(p+1)} + \ldots$$
(11)

and when $k = 7$ equation (11) is reduced to

$$T_{n+7} = c_0 y_n + c_1 h y'_n + c_2 h^2 y''_n + c_3 h^3 y'''_n + \ldots + c_8 h^8 y_n^{(8)} + c_9 h^9 y_n^{(9)} + 0(h^{10})$$
(12)

Definition 1

Equation (12) and the associated continuous linear multistep step method presented in table 1 and 2 are said to be of order p if $c_0 = c_1 = c_2 = \ldots = c_p$ *and* $c_{p+1} \neq 0$, c_{p+1} *is called the error constant and the local truncation error given by*

$$t_{n+k} = c_{p+1} h^{(p+1)} y_n^{(p+1)} + 0(h^{p+2}) \qquad (13)$$

Using (12) and definition (1), we compute the order and error constant for the first equation in table 1 and 2, as

$$c_0 = \frac{1}{840}\left(-1 + 14 - 126 - 525 + 525 + 126 - 14 + 1\right) = 0$$

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$$c_1 = \frac{1}{840}(14 - 252 - 1575 + 2100 + 630 - 84 + 7) - \left(\frac{1}{2} + \frac{1}{2}\right) = 0$$

$$c_2 = \frac{1}{1680}(14 - 504 - 4725 + 8400 + 3150 - 504 + 49) - \left(\frac{3}{2} + \frac{4}{2}\right) = 0$$

$$c_3 = \frac{1}{5040}(14 - 1008 - 14175 + 33600 + 15750 - 3024 + 343) - \left(\frac{9}{4} + \frac{16}{4}\right) = 0$$

$$c_4 = \frac{1}{20160}(14 - 2016 - 42525 + 134400 + 78750 - 18144 + 2401)$$
$$-\left(\frac{27}{12} + \frac{64}{12}\right) = 0$$

$$c_5 = \frac{1}{100800}(14 - 4032 - 127575 + 537600 + 393750 - 108864 + 16807)$$
$$-\left(\frac{81}{48} + \frac{256}{48}\right) = 0$$

$$c_6 = \frac{1}{604800}(14 - 8064 - 382725 + 2150400 + 1968750 - 653184 + 117649)$$
$$-\left(\frac{243}{240} + \frac{1024}{240}\right) = 0$$

$$c_7 = \frac{1}{4233600}(14 - 16128 - 1148175 + 8601600 + 9843750 - 3919104 + 823543)$$
$$-\left(\frac{729}{1004} + \frac{4096}{1004}\right) = 0$$

$$c_8 = \frac{1}{33868800}(14 - 32256 - 3444525 + 34406400 + 49218750 - 23514624 + 5764801)$$
$$-\left(\frac{2187}{10080} + \frac{16384}{10080}\right) = 0$$

$$c_9 = \frac{1}{304819200}(14 - 64512 - 10333575 + 137625600 + 246093750 - 141087744 + 40353607)$$
$$-\left(\frac{6561}{80640} + \frac{65536}{80640}\right) \neq 0, c_9 = \frac{1}{5040}$$

Using this approach, we can compute the order and error constants for the remaining set of equations in table 1 and 2. Thus, giving us a uniform order which is presented as

$$p = (8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8)^T$$

$$c_{p+1} = \left(\frac{1}{5040} \quad \frac{667}{3051720} \quad \frac{1}{7056} \quad \frac{1}{14112} \quad -\frac{1}{17640} \quad -\frac{1}{7056} \quad -\frac{1}{5292}\right)^T$$

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Consistency

Definition 2

The LMM presented in table 1 and 2 is said to be consistent if it satisfies the following conditions

The order $p \geq 1$

$$\sum_{j=o}^{k} \alpha_j = 0$$

$\rho'(1) = \sigma(1),$ *where,* $\rho(r)$ *and* $\sigma(r)$ *are respectively the first and second characteristic polynomials of the individual equations.*

Using the second equation in table 1 and 2 for illustration purpose, we get

$$p = \frac{667}{3051720} \geq 1$$

$$\sum_{j=o}^{k} \alpha_j = \left(-\frac{91}{72660} + \frac{1264}{72660} - \frac{11250}{72660} - \frac{45300}{72660} + \frac{46175}{72660} + \frac{10116}{72660} - \frac{914}{72660} \right) = 0$$

$$\rho(r) = -\frac{914}{72660} r^6 + \frac{10116}{72660} r^5 + \frac{46175}{72660} r^4 - \frac{45300}{72660} r^3 - \frac{11250}{72660} r^2 + \frac{1264}{72660} r - \frac{91}{72660}$$

and

$$\sigma(r) = -\frac{1}{2422} r^7 + \frac{1194}{2422} r^4 + \frac{1229}{2422} r^3$$

then,

$$\rho'(1) = -\frac{914}{12110} + \frac{10116}{14532} + \frac{46175}{18165} - \frac{45300}{24220} - \frac{11250}{36330} + \frac{1264}{72660} = 1$$

and

$$\sigma(1) = -\frac{1}{2422} + \frac{1194}{2422} + \frac{1229}{2422} = 1$$

thus,

$$\rho'(1) = \sigma(1) = 1$$

We have therefore established that the method is consistent.

Zero-Stability

Definition 3

The block method (table 1 and 2) is said to be zero-stable, if the roots $\lambda_i, i = 1,2,\ldots,k$ *of the first characteristic polynomial* $\rho(\lambda)$ *defined by* $\rho(\lambda) = \det\left(\lambda A^{(0)} - \phi\right)$ *satisfies* $|\lambda_i| \leq 1$ *and every root satisfying* $|\lambda_i| \leq 1$ *have multiplicity not exceeding the order of the differential equation. Moreover, as* $h \to 0$, $\rho(\lambda) = \lambda^{r-\upsilon}(\lambda - 1)^{\upsilon}$ *where* υ *is the order of the differential equation,* r *is the order of the matrices* $A^{(0)}$ *and* ϕ (Awoyemi *et al*., 2007).

Thus, our block method yields

$$\rho(\lambda)=\det\left|\lambda\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right|=0 \qquad (14)$$

$\rho(\lambda)=\lambda^6(\lambda-1)=0,\ \lambda_1=\lambda_2=...=\lambda_6=0,\ \lambda_7=1$. Following definition 3 and from (14), our eight order block ETR$_2$s method has been shown to be zero-stable.

Convergence

Definition 4

The necessary condition for an linear multistep method to be convergent are that it be consistent and zero-stable (Dahlquist, 1956; Henrici, 1962; Fatunla, 1988; Lambert, 1991).

Since the newly developed eight order block ETR$_2$s method has been shown to be consistent and zero- stable, it is therefore convergent (definition 4).

STABILITY REGION OF THE BLOCK METHOD

The main difficulty associated with stiff equations is that even though the component of the true solutions corresponding to some eigenvalues that may be becoming negligible, the restriction on the stepsize imposed by the numerical stability of the method requires that $|h\lambda|$ remain small throughout the range of integration. So a suitable formula for stiff equations would be the one that would not require that $|h\lambda|$ remains small [16].

Definition 5

The stability region R associated with a multistep formula is defined as the set

$R=\{h\lambda :$ a numerical formula applied to $y'=\lambda y,\ y(x_0)=y_0$, with constant stepsize $h>0$, produce a sequence (y_0) satisfying that $y_n\to 0$ as $n\to\infty\}$

Definition 6

A linear multistep method is said to be $A(\alpha)$-stable, $\alpha\in(0,\frac{\pi}{2})$ if it's region of absolute stability contains the infinite wedge $W_\alpha=\{h\lambda|-\alpha<\pi-\arg h\lambda<\alpha\}$.

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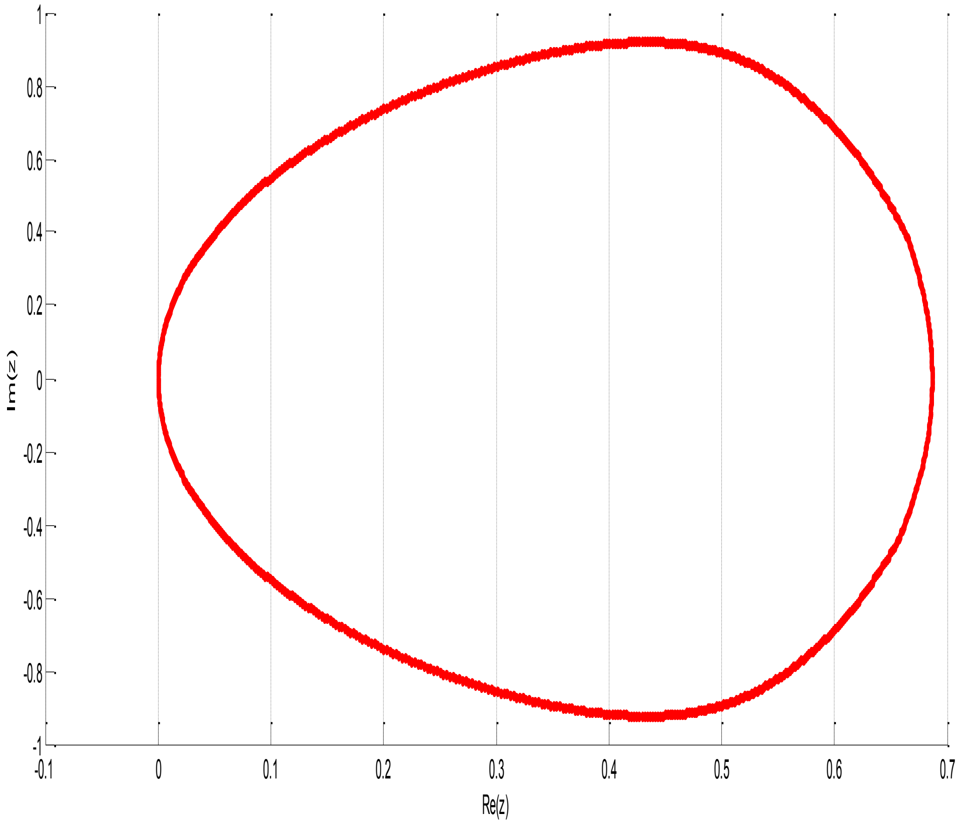

Figure 1: Stability region of the eighth order block ETR_2s method

Definition 7

A linear multistep method is said to be A-stable, if it's region of absolute stability contains the whole of the left hand complex plane.

NUMERICAL EXPERIMENT

In order to assess the performance of our block ETR_2s method, it was applied to the integration of some specific systems of first order ordinary differential equations.

Ricatti equation

Let us consider the following Ricatti equation (Abramowitz and stegun, 1972)

$$y' = -2 - y + y^2$$

with initial value $y_0 = 1.8$, then the theoretical solution is given by

$$y(t) = 2 - \frac{3}{1 + 14e^{-3t}}$$

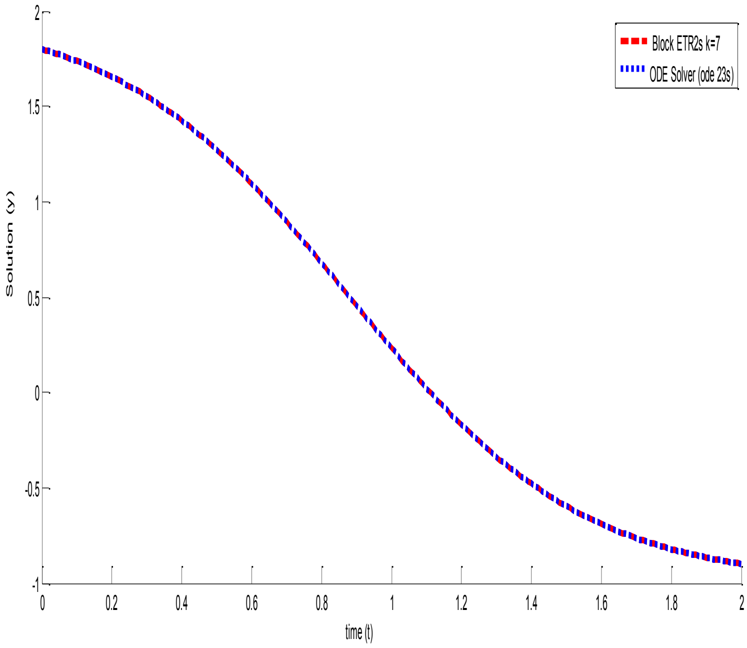

Figure 2: Solution Curve for problem 5.1

5.2 A numerical example solved by Mehdizadeh *et al*., 2012 which was reported by Cash, 1981**.**

$$y_1' = -\alpha y_1 - \beta y_2 + (\alpha + \beta - 1)e^{-x}$$

$$y_2' = \beta y_1 - \alpha y_2 - (\alpha - \beta - 1)e^{-x}$$

with initial value $y(0) = (1,1)^T$. The required solution is $y_1(x) = y_2(x) = e^{-x}$. Results are obtained when $\alpha = 1,\ \beta = 30$ and $h = 0.09$ are chosen.

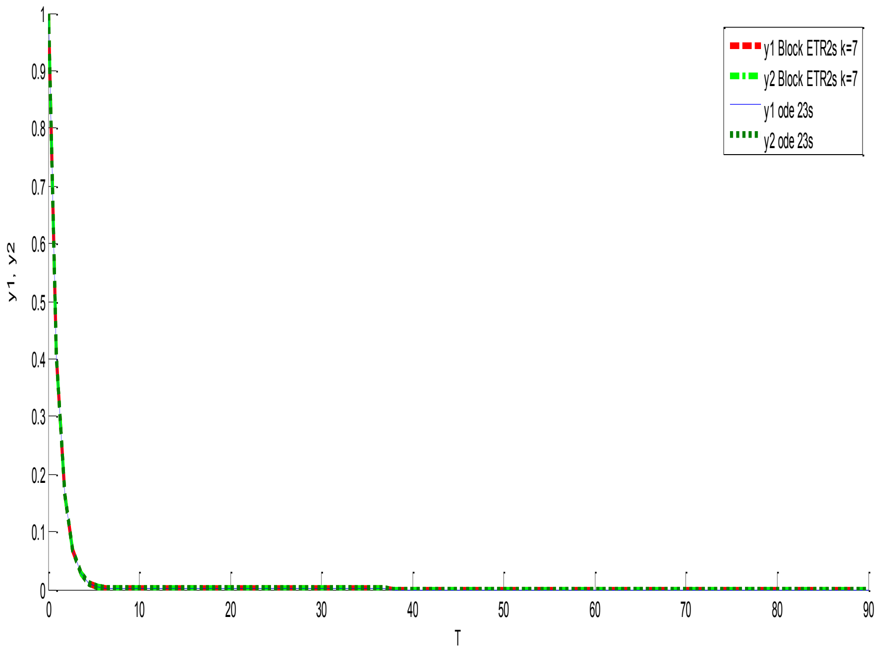

Figure 3: Solution Curve for problem 5.2

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5.3: Consider the Lorenz Equations reported by Ya Yan Lu

$y_1' = 10(y_2 - y_1)$

$y_2' = -y_1 y_3 + 28y_1 - y_2$

$y_3' = y_1 y_2 - \frac{8}{3} y_3$

$y_1(0) = -11.3360\ ,\ y_2(0) = -16.0335,\ y_3(0) = 24.4450,\ 0 \le x \le 40,\ h = 0.02$

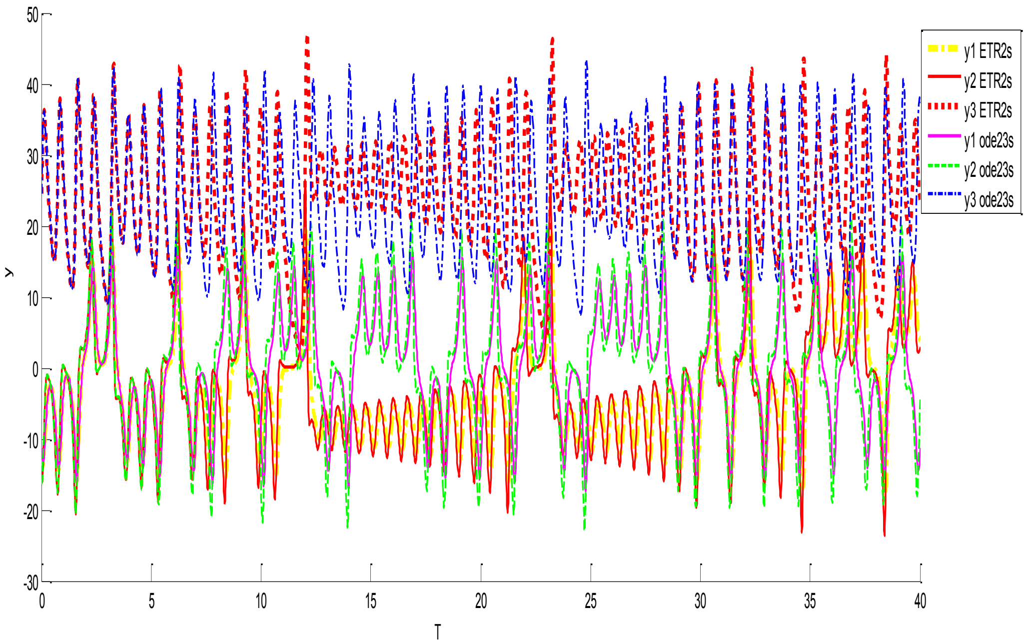

Figure 4: Solution Curve for problem 5.3

5.4 We consider the given linear stiff system on the range $0 \le x \le 20$ which is highly oscillatory with all of its eigenvalues near the imaginary axis.

$y_1' = -10y_1 + 50y_2 \qquad y_1(0) = 1$

$y_2' = -50y_1 - 10y_2 \qquad y_2(0) = 1$

$y_3' = -40y_3 + 200y_4 \qquad y_3(0) = 0$

$y_4' = -200y_3 - 40y_4 \qquad y_1(0) = 1$

$y_5' = -0.2y_5 - 2y_6 \qquad y_2(0) = 0$

$y_6' = -2y_5 - 0.2y_6 \qquad y_3(0) = 1$

This problem has also been solved by Rockswold [17]

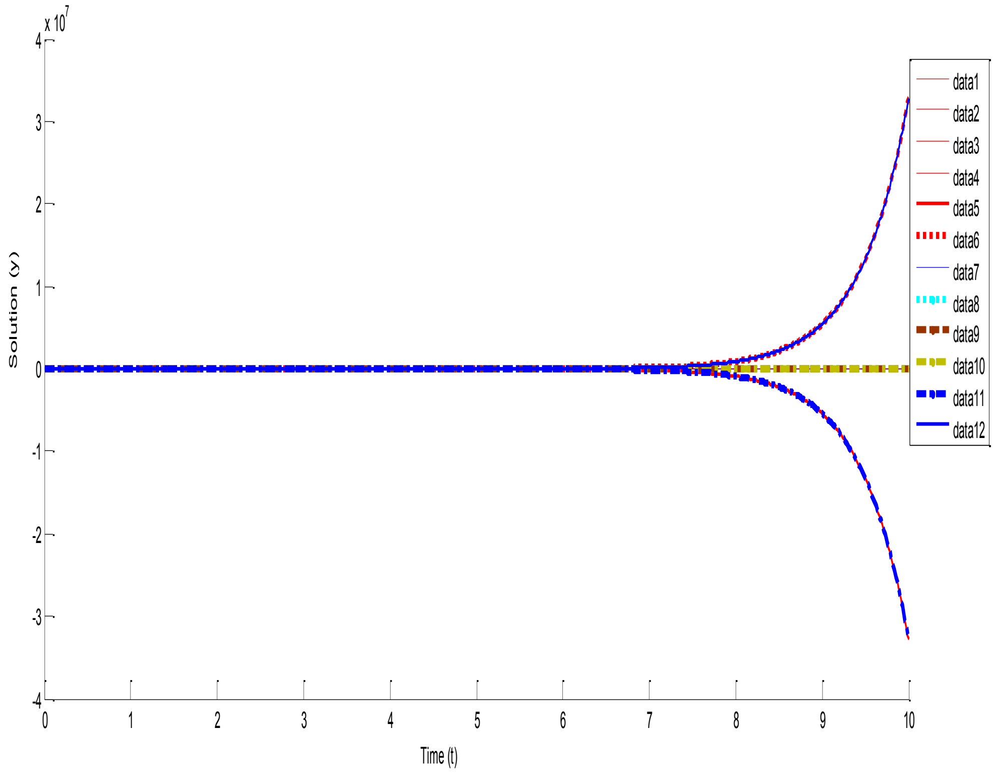

Figure 5: Solution Curve for problem 5.4

CONCLUSION

In this paper, we developed an eight order block Extended Trapezoidal Rule of Second kind (ETR_2s). Our block method is shown to be A-Stable (Fig. 1) and so very suitable for stiff system of ordinary differential equations. Consequently, Our newly derived eight order block ETR_2s is shown to compete favorably with the well known MatLab Ode solver Ode 23s (fig. 2-5).

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