REGION OF ABSOLUTE STABILITY OF AN ORDER 19 RATIONAL INTEGRATOR

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ABSTRACT

In this research, the Region of Absolute Stability (RAS) of an order 19 - rational integrator is presented. The analysis covers the coefficient parameters of the $u - v$ form and (R, θ)polar form of the stability function.The polar form is conducted and studiedthrough polar curves that enable us to determine the RAS of the integrator. We carried out relevant tests experimentally.

Keywords and Phrases: Rational integrator, stiff, singular and oscillatory problems, stability function and Region of Absolute Stability (RAS)

INTRODUCTION

Preliminaries

We consider here, the initial value problem given by

$$y' = f(x,y), \qquad y(x_0) = y_0, \qquad x \in [a,b], y \in R \tag{1.1}$$

whose solution may be stiff, singular or oscillatory. In this paper, we are concerned with the Region of Absolute Stability (RAS)of the 19thorder rational integrator. Stability concept is a very important aspect of numerical integration; for example, according to Lambert [1], zero–stability controls the manner in which the local truncation error is propagated in the limit as $h \to 0$, which in essence implies the existence of a positive h^* such that for $h \in (0, h^*)$, stable propagation of error will occur.

One major area of concern is the stability of result of poles of singular problems. To this extent, the rational function, $\frac{P_m(x)}{Q_n(x)}$ takes the singularities of the function to be approximated such that the singularity of $f(x) \simeq \frac{P_m(x)}{Q_n(x)}$ is the zero of $Q_n(x)$. The importance of this area of work is exemplified by the several authors who have worked in this area;these include Fatunla [2,3,4], Aashikpelokhai [5], Otunta and Ikhile [6], Onianwa and Aashikpelokhai [7], Aashikpelokhai and Onianwa [8], Agbeboh and Aashikpelokhai [9], Abhulimen and Ukpebor [10], Elakhe and Aashikpelokhai [11], Sunday and Odekunle [12], Ukpebor and Aashikpelokhai [13].

Generally, when we study the Region of Absolute Stability, RAS great use is made of $|S(\bar{h})| = 1$ in the case of one – step method, where $S(\bar{h})$ is the stability function of the method. The central idea is that $|S(\bar{h})| = 1$ defines the boundary between the RAS and the Region of Instability (RI) in the complex plane.

From Aashikpelokhai [5] our rational integrator is given by

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$$y_{n+1} = \frac{\sum_{i=0}^{k-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^{k} q_i x_{n+1}^i} \qquad (1.2)$$

To investigate the stability properties of the rational integrator for any k, Aashikpelokhai [5] used the matrix equation which is derived from the result of matching the Taylor series of $y(x_{n+1})$ and that of the approximated value y_{n+1} to have the matrix form of the stability expression and then proceeded to useGaussian elimination to solve for the exact$q_i(\bar{h})$, $i = 1(1)k$. These are then used to obtain the exact $p_i(\bar{h})$, $i = 1(1)k - 1$, which we now write as

$$q_i(\bar{h}, k) = \frac{(-1)^i (2k-1-i)! \binom{k}{i} \bar{h}^i}{(2k-1)!\, x_{n+1}^i}, \qquad i = 0(1)k \qquad (1.3)$$

and

$$p_i(\bar{h}, k) = \frac{(2k-1-i)! \binom{k-1}{i} \bar{h}^i}{(2k-1)!\, x_{n+1}^i} y_n, \qquad i = 0(1)k-1 \qquad (1.4)$$

Simply writing $q_i(\bar{h}, k)$ and $p_i(\bar{h}, k)$ as q_i and p_i respectively, we have

$$q_i x_{n+1}^i = \frac{(-1)^i (2k-1-i)! \binom{k}{i} \bar{h}^i}{(2k-1)!}, \qquad i = 0(1)k \qquad (1.5)$$

and

$$p_i x_{n+1}^i = \frac{(2k-1-i)! \binom{k-1}{i} \bar{h}^i}{(2k-1)!} y_n, \qquad i = 0(1)k-1 \qquad (1.6)$$

Using (1.4) and (1.5) in (1.1), Aashikpelokhai [5] obtained

$$y_{n+1} = \frac{\sum_{i=0}^{k-1} (2k-1-i)! \binom{k-1}{i} \bar{h}^i}{\sum_{i=0}^{k} (-1)^i (2k-1-i)! \binom{k}{i} \bar{h}^i} y_n \qquad (1.7)$$

The stability function $S(\bar{h})$ of any arbitrary positive integer k is therefore given by

$$S(\bar{h}) = \frac{y_{n+1}}{y_n} \qquad (1.8)$$

$$i.e. \quad S(\bar{h}) = \frac{\sum_{i=0}^{k-1} (2k-1-i)! \binom{k-1}{i} \bar{h}^i}{\sum_{i=0}^{k} (-1)^i (2k-1-i)! \binom{k}{i} \bar{h}^i} \qquad (1.9)$$

STABILITY CONSIDERATION

Stability Function

Derivation

For $k = 10$, the rational integrator is given by

$$y_{n+1} = \frac{\sum_{i=0}^{9} p_i x_{n+1}^i}{\sum_{i=0}^{10} q_i x_{n+1}^i} \qquad (2.1.1)$$

where our integrator parameters, $p_i's$ and $q_i's$ are given by

$$p_j = \sum_{i=1}^{j} \frac{h^{j+1-i} y_n^{(j+1-i)} q_{i-1}}{(j+1-i)!\, x_n^{j+1-i}} + y_n q_j\,, \qquad j = 1(1)k-1 \qquad (2.1.2)$$

with the q_j obtained from the Simultaneous Linear Equation (SLE),$\underline{\underline{s}}\underline{q} = \underline{b}$ where

$$s_{ij} = \frac{h^{2k-i-j} y_n^{(2k-i-j)}}{(2k-i-j)!\, x_{n+1}^{2k-i-j}}, \quad i,j = 1(1)k \qquad (2.1.3)$$

defines the entries of the matrix s, the vector q is given by

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_k \end{bmatrix} (2.1.4)$$

and the vector b is given by

$$b_i = \frac{h^{2k-1} y_n^{(2k-i)}}{(2k-i)!\, x_{n+1}^{2k-i}} \qquad i = 1(1)(10) \qquad (2.1.5)$$

To derive the stability function we employ the linearized stability relations $y^{(1)} = \lambda y$ and $\overline{h} = \lambda h$ where h is the usual mesh size.

The application is done by employing the $y^{(1)} = \lambda y$ and $\overline{h} = \lambda h$ on the SLE $\underline{\underline{s}}\underline{q} = \underline{b}$ and then solve the resulting equation, now in $\overline{h}$ for q_j. The solution produce q_j below and by the p_j, we get the p_j below

$$q_0 = 1, \qquad q_1 = \frac{18!\,10\overline{h}}{19!\,x_{n+1}}, \qquad q_2 = \frac{17!\,45\overline{h}^2}{19!\,x_{n+1}^2}, \qquad q_3 = \frac{16!\,120\overline{h}^3}{19!\,x_{n+1}^3},$$

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$$q_4 = \frac{15!\,210\bar{h}^4}{19!\,x_{n+1}^4}, \qquad q_5 = \frac{14!\,252\bar{h}^5}{19!\,x_{n+1}^5}, \qquad q_6 = \frac{13!\,210\bar{h}^6}{19!\,x_{n+1}^6}, \qquad q_7 = \frac{12!\,120\bar{h}^7}{19!\,x_{n+1}^7},$$

$$q_8 = \frac{11!\,45\bar{h}^8}{19!\,x_{n+1}^8}, \qquad q_9 = \frac{10!\,10\bar{h}^9}{19!\,x_{n+1}^9}, \qquad q_{10} = \frac{9!\,\bar{h}^{10}}{19!\,x_{n+1}^{10}}$$

(2.1.6)

and

$$p_0 = y_n, \qquad p_1 = \frac{18!\,9\bar{h}}{19!\,x_{n+1}}y_n, \qquad p_2 = \frac{17!\,36\bar{h}^2}{19!\,x_{n+1}^2}y_n, \qquad p_3 = \frac{16!\,42\bar{h}^3}{19!\,x_{n+1}^3}y_n,$$

$$p_4 = \frac{15!\,126\bar{h}^4}{19!\,x_{n+1}^4}y_n, \qquad p_5 = \frac{14!\,126\bar{h}^5}{19!\,x_{n+1}^5}y_n, \qquad p_6 = \frac{13!\,84\bar{h}^6}{19!\,x_{n+1}^6}y_n, \qquad p_7 = \frac{12!\,36\bar{h}^7}{19!\,x_{n+1}^7}y_n,$$

$$p_8 = \frac{11!\,9\bar{h}^8}{19!\,x_{n+1}^8}y_n, \qquad p_9 = \frac{10!\,\bar{h}^9}{19!\,x_{n+1}^9}y_n$$

(2.1.7)

Substituting (2.1.2) and (2.1.3) in (2.1.1) we have the stability function

$$S(\bar{h}) = \frac{\sum_{i=0}^{9}(19-i)!\binom{9}{i}\bar{h}^i}{\sum_{i=0}^{10}(-1)^i(19-i)!\binom{10}{i}\bar{h}^i} \tag{2.1.8}$$

where

$$S(\bar{h}) = \frac{y_{n+1}}{y_n} \tag{2.1.9}$$

The Stability Function, $S(u,v)$ And Its Coefficient Distribution

By writing $\bar{h} = u + iv,\ i = \sqrt{-1},$ (1.9) becomes

$$S(u,v) = \frac{\sum_{r=0}^{9}(19-r)!\binom{9}{r}(u+iv)^r}{\sum_{r=0}^{10}(-1)^r(19-r)!\binom{10}{r}(u+iv)^r} \tag{2.2.1}$$

For ease of work we let

$$S(u,v) = \frac{\phi(u,v)}{\psi(u,v)} \qquad (2.2.2)$$

where

$$\phi(u,v) = \sum_{r=0}^{9}(19 - r)!\binom{9}{r}(u+iv)^r \qquad (2.2.3)$$

$$\psi(u,v) = \sum_{r=0}^{10}(-1)^r(19 - r)!\binom{10}{r}(u+iv)^r \qquad (2.2.4)$$

Next we are collecting components by setting $\phi(u,v) = A(u,v) + iB(u,v)$ and $\psi(u,v) = C(u,v) + iD(u,v)$ leading us to the following results

$$|S(u,v)| \leq 1, \qquad whenver\ u < 0$$

of various k.

$$|S(u,v)| \leq 1 \Leftrightarrow \frac{\phi(u,v)}{\psi(u,v)} \leq 1$$

$$\Leftrightarrow |\phi(u,v)|^2 \leq |\psi(u,v)|^2$$

$$\Leftrightarrow |\phi(u,v)|^2 - |\psi(u,v)|^2 \leq 0$$

$$\Leftrightarrow [A^2(u,v) + B^2(u,v)] - [C^2(u,v) + D^2(u,v)] \leq 0 \qquad (2.2.5)$$

We now herein define $F(u,v) = [A^2 + B^2] - [C^2 + D^2] = [A^2 - C^2] + [B^2 - D^2]$ (2.2.6)

Hence

$$|S(u,v)| \leq 1 \Leftrightarrow F(u,v) \leq 0 \qquad (2.2.7)$$

Where $A(u,v), B(u,v), C(u,v), D(u,v)$ through the identity

$\phi(u,v) = A + iB, \psi(u,v) = C + iD$ leads to

$$\phi(u,v) = \sum_{r=0}^{9}(2 \cdot 10 - 1 - r)!\binom{9}{r}(u+iv)^r \equiv A(u,v) + iB(u,v) \qquad (2.2.8)$$

$i.e. A(u,v) = Re\{\phi(u,v)\}$ while $B(u,v) = Im\{\phi(u,v)\}$

$$= T_0 + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9$$

Where T_r refers to term $(r+1)$

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Let $E_r = (2\cdot 10-1-r)!\binom{9}{r}$, then

$$\phi(u,v)=\sum_{r=0}^{9} E_r(u+iv)^r$$
$$= E_0 + E_1(u+iv) + E_2(u+iv)^2 + E_3(u+iv)^3 + \cdots + E_9(u+iv)^9 \quad (2.2.9)$$

By (2.2.8) and (2.2.9) we have $A_r = E_r(u+iv)^r,\ 0\le r\le 9$

Hence; $T_0 = E_0(u+iv)^0 = (2\cdot 10-1-0)!\binom{9}{0}$

$$T_1 = E_1(u+iv)^1 = (2\cdot 10-1-1)!\binom{9}{1}u + i(2\cdot 10-1-1)!\binom{9}{1}v$$

$$T_2 = E_2(u+iv)^2 = (2\cdot 10-1-2)!\binom{9}{2}(u^2-v^2) + i(2\cdot 10-1-2)!\binom{9}{2}2uv$$

$$T_3 = E_3(u+iv)^3 = (2\cdot 10-1-3)!\binom{9}{3}(u^3-3uv^2) + i(2\cdot 10-1-3)!\binom{9}{3}(3u^2v-v^3)$$

$$T_4 = E_4(u+iv)^4 = (2\cdot 10-1-4)!\binom{9}{4}(u^4-6u^2v^2+v^4) + i(2\cdot 10-1-4)!\binom{9}{4}(4u^3v - 4uv^3)$$

$$T_5 = E_5(u+iv)^5 = (2\cdot 10-1-5)!\binom{9}{5}(u^5-10u^3v^2+5uv^4) + i(2\cdot 10-1-5)!\binom{9}{5}(5u^4v-10u^2v^3+v^5)$$

$$T_6 = (u+iv)^6 = (2\cdot 10-1-6)!\binom{9}{6}(u^6-15u^4v^2+15u^2v^4-v^6) + i(2\cdot 10-1-6)!\binom{9}{6}(6u^5v-20u^3v^3+6uv^5)$$

$$T_7 = E_7(u+iv)^7 = (2\cdot 10-1-7)!\binom{9}{7}(u^7-21u^5v^2+35u^3v^4-7uv^6) + i(2\cdot 10-1-7)!\binom{9}{7}(7u^6v-35u^4v^3+21u^2v^5-v^7)$$

$$T_8 = E_8(u+iv)^8 = (2\cdot 10-1-8)!\binom{9}{8}(u^8-28u^6v^2+70u^4v^4-28u^2v^6+v^8) + i(2\cdot 10-1-8)!\binom{9}{8}(8u^7v-56u^5v^3+56u^3v^5-8uv^7)$$

$$T_9 = E_9(u+iv)^9 = (2\cdot 10-1-9)!\binom{9}{9}(u^9-36u^7v^2+126u^5v^4-84u^3v^6 + 9uv^8)$$

$$+\, i(2\cdot 10-1-9)!\binom{9}{9}(9u^8v-84u^6v^3+126u^4v^5-36u^2v^7+v^9)$$

Recalling the real part of $\phi(u,v)$ as$A(u,v)$ and the imaginary part as$B(u,v)$. Thus we have

$$\begin{aligned}
A(u,v) = \Big[&(2\cdot 10-1-0)!\binom{9}{0}+(2\cdot 10-1-1)!\binom{9}{1}u+(2\cdot 10-1-2)!\binom{9}{2}u^2\\
&-(2\cdot 10-1-2)!\binom{9}{2}v^2+(2\cdot 10-1-3)!\binom{9}{3}u^3-(2\cdot 10-1-3)!\binom{9}{3}3uv^2\\
&+(2\cdot 10-1-4)!\binom{9}{4}u^4-(2\cdot 10-1-4)!\binom{9}{4}6u^2v^2\\
&\qquad+(2\cdot 10-1-4)!\binom{9}{4}v^4\\
&+(2\cdot 10-1-5)!\binom{9}{5}u^5-(2\cdot 10-1-5)!\binom{9}{5}10u^3v^2\\
&\qquad+(2\cdot 10-1-5)!\binom{9}{5}5uv^4\\
&+(2\cdot 10-1-6)!\binom{9}{6}u^6-(2\cdot 10-1-6)!\binom{9}{6}15u^4v^2\\
&\qquad+(2\cdot 10-1-6)!\binom{9}{6}15u^2v^4\\
&-(2\cdot 10-1-6)!\binom{9}{6}v^6+(2\cdot 10-1-7)!\binom{9}{7}u^7\\
&\qquad-(2\cdot 10-1-7)!\binom{9}{7}21u^5v^2\\
&+(2\cdot 10-1-7)!\binom{9}{7}35u^3v^4-(2\cdot 10-1-7)!\binom{9}{7}7uv^6\\
&\qquad+(2\cdot 10-1-8)!\binom{9}{8}u^8\\
&-(2\cdot 10-1-8)!\binom{9}{8}28u^6v^2+(2\cdot 10-1-8)!\binom{9}{8}70u^4v^4\\
&\qquad-(2\cdot 10-1-8)!\binom{9}{8}28u^2v^6\\
&+(2\cdot 10-1-8)!\binom{9}{8}v^8+(2\cdot 10-1-9)!\binom{9}{9}u^9\\
&\qquad-(2\cdot 10-1-9)!\binom{9}{9}36u^7v^2\\
&+(2\cdot 10-1-9)!\binom{9}{9}126u^5v^4-(2\cdot 10-1-9)!\binom{9}{9}84u^3v^6\\
&\qquad+(2\cdot 10-1-9)!\binom{9}{9}9uv^8\Big]
\end{aligned} \tag{2.2.10}$$

$$\begin{aligned}
B(u,v) = \Big[&(2\cdot 10-1-1)!\binom{9}{1}v+(2\cdot 10-1-2)!\binom{9}{2}2uv\\
&\quad+(2\cdot 10-1-3)!\binom{9}{3}3u^2v\\
&-(2\cdot 10-1-3)!\binom{9}{3}v^3+(2\cdot 10-1-4)!\binom{9}{4}4u^3v\\
&\qquad-(2\cdot 10-1-4)!\binom{9}{4}4uv^3
\end{aligned}$$

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$$+(2\cdot10-1-5)!\binom{9}{5}5u^4v-(2\cdot10-1-5)!\binom{9}{5}10u^2v^3$$
$$+(2\cdot10-1-5)!\binom{9}{5}v^5$$
$$+(2\cdot10-1-6)!\binom{9}{6}6u^5v-(2\cdot10-1-6)!\binom{9}{6}20u^3v^3$$
$$+(2\cdot10-1-6)!\binom{9}{6}6uv^5$$
$$+(2\cdot10-1-7)!\binom{9}{7}7u^6v-(2\cdot10-1-7)!\binom{9}{7}35u^4v^3$$
$$+(2\cdot10-1-7)!\binom{9}{7}21u^2v^5$$
$$-(2\cdot10-1-7)!\binom{9}{7}v^7+(2\cdot10-1-8)!\binom{9}{8}8u^7v$$
$$-(2\cdot10-1-8)!\binom{9}{8}56u^5v^3$$
$$+(2\cdot10-1-8)!\binom{9}{8}56u^3v^5-(2\cdot10-1-8)!\binom{9}{8}8uv^7$$
$$+(2\cdot10-1-9)!\binom{9}{9}9u^8v$$
$$-(2\cdot10-1-9)!\binom{9}{9}84u^6v^3+(2\cdot10-1-9)!\binom{9}{9}126u^4v^5$$
$$-(2\cdot10-1-9)!\binom{9}{9}36u^2v^7$$
$$+\ (2\cdot10-1-9)!\binom{9}{9}v^9\Big]$$

(2.2.11)

Considering the denominator,

$$\psi(u,v)=\sum_{r=0}^{10}(-1)^r(2.10-1-r)!\binom{10}{r}(u+iv)^r$$
$$\equiv C(u,v)+iD(u,v) \qquad (2.2.12)$$

$i.e. C(u,v)=Re\{\psi(u,v)\}$ while $D(u,v)=Im\{\psi(u,v)\}$

$$=R_0+R_1+R_2+R_3+R_4+R_5+R_6+R_7+R_8+R_9+R_{10} \qquad (2.2.13)$$

Let $P_r=(-1)^r(2\cdot10-1-r)!\binom{10}{r}$, so that

$$\psi(u,v)=\sum_{r=0}^{10}P_r\,(u+iv)^r$$
$$=P_0+P_1(u+iv)+P_2(u+iv)^2+P_3(u+iv)^3+\cdots+P_{10}(u+iv)^{10} \quad (2.2.14)$$

By (2.2.13) and (2.2.14) we have $R_r=P_r(u+iv)^r,\ 0\le r\le10$

Hence $R_0=P_0=(2\cdot10-1-0)!\binom{10}{0}$

$$R_1=P_1(u+iv)=-(2\cdot10-1-1)!\binom{10}{1}u \qquad -i(2\cdot10-1-1)!\binom{10}{1}v$$

$$R_2 = P_2(u+iv)^2$$
$$= (2\cdot 10-1-2)!\binom{10}{2}(u^2-v^2)$$
$$+ i(2\cdot 10-1-2)!\binom{10}{2}2uv$$

$$R_3 = P_3(u+iv)^3$$
$$= -(2\cdot 10-1-3)!\binom{10}{3}(u^3-3uv^2)$$
$$- i(2\cdot 10-1-3)!\binom{10}{3}(3u^2v-v^3)$$

$$R_4 = P_4(u+iv)^4 = (2\cdot 10-1-4)!\binom{10}{4}(u^4-6u^2v^2+v^4)$$
$$+ i(2\cdot 10-1-4)!\binom{10}{4}(4u^3v-4uv^3)$$

$$R_5 = P_5(u+iv)^5 = -(2\cdot 10-1-5)!\binom{10}{5}(u^5-10u^3v^2+5uv^4)$$
$$- i(2\cdot 10-1-5)!\binom{10}{5}(5u^4v-10u^2v^3+v^5)$$

$$R_6 = P_6(u+iv)^6 = (2\cdot 10-1-6)!\binom{10}{6}(u^6-15u^4v^2+15u^2v^4-v^6)$$
$$+ i(2\cdot 10-1-6)!\binom{10}{6}(6u^5v-20u^3v^3+6uv^5)$$

$$R_7 = P_7(u+iv)^7 = -(2\cdot 10-1-7)!\binom{10}{7}(u^7-21u^5v^2+35u^3v^4-7uv^6)$$
$$- i(2\cdot 10-1-7)!\binom{10}{7}(7u^6v-35u^4v^3+21u^2v^5-v^7)$$

$$R_8 = P_8(u+iv)^8$$
$$= (2\cdot 10-1-8)!\binom{10}{8}(u^8-28u^6v^2+70u^4v^4-28u^2v^6+v^8)$$
$$+ i(2\cdot 10-1-8)!\binom{10}{8}(8u^7v-56u^5v^3+56u^3v^5-8uv^7)$$

$$R_9 = P_9(u+iv)^9$$
$$= -(2\cdot 10-1-9)!\binom{10}{9}(u^9-36u^7v^2+126u^5v^4-84u^3v^6$$
$$+9uv^8)$$
$$- i(2\cdot 10-1-9)!\binom{10}{9}(9u^8v-84u^6v^3+126u^4v^5-36u^2v^7$$
$$+v^9)$$

$$R_{10} = P_{10}(u+iv)^{10}$$
$$= (2\cdot 10-1-10)!\binom{10}{10}(u^{10}-45u^8v^2+210u^6v^4-210u^4v^6$$
$$+45u^2v^8-v^{10})$$
$$+ i(2\cdot 10-1-10)!\binom{10}{10}(10u^9v-120u^7v^3+252u^5v^5-120u^3v^7+10uv^9)$$

Recalling the real part of $\psi(u,v)$ as $C(u,v)$ and the imaginary part as $D(u,v)$, thus we have

$$C(u,v) = (2\cdot 10-1-0)!\binom{10}{0} - (2\cdot 10-1-1)!\binom{10}{1}u + (2\cdot 10-1-2)!\binom{10}{2}u^2$$

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$$
\begin{aligned}
&- (2\cdot 10-1-2)!\binom{10}{2}v^2 - (2\cdot 10-1-3)!\binom{10}{3}u^3 \\
&\qquad + (2\cdot 10-1-3)!\binom{10}{3}3uv^2 \\
&+ (2\cdot 10-1-4)!\binom{10}{4}u^4 - (2\cdot 10-1-4)!\binom{10}{4}6u^2v^2 \\
&\qquad + (2\cdot 10-1-4)!\binom{10}{4}v^4 \\
&- (2\cdot 10-1-5)!\binom{10}{5}u^5 + (2\cdot 10-1-5)!\binom{10}{5}10u^3v^2 \\
&\qquad - (2\cdot 10-1-5)!\binom{10}{5}5uv^4 \\
&+ (2\cdot 10-1-6)!\binom{10}{6}u^6 - (2\cdot 10-1-6)!\binom{10}{6}15u^4v^2 \\
&\qquad + (2\cdot 10-1-6)!\binom{10}{6}15u^2v^4 \\
&- (2\cdot 10-1-6)!\binom{10}{6}v^6 - (2\cdot 10-1-7)!\binom{10}{7}u^7 \\
&\qquad + (2\cdot 10-1-7)!\binom{10}{7}21u^5v^2 \\
&-(2\cdot 10-1-7)!\binom{10}{7}35u^3v^4 + (2\cdot 10-1-7)!\binom{10}{7}7uv^6 \\
&\qquad + (2\cdot 10-1-8)!\binom{10}{8}u^8 \\
&- (2\cdot 10-1-8)!\binom{10}{8}28u^6v^2 + (2\cdot 10-1-8)!\binom{10}{8}70u^4v^4 \\
&\qquad - (2\cdot 10-1-8)!\binom{10}{8}28u^2v^6 \\
&+ (2\cdot 10-1-8)!\binom{10}{8}v^8 - (2\cdot 10-1-9)!\binom{10}{9}u^9 \\
&\qquad + (2\cdot 10-1-9)!\binom{10}{9}36u^7v^2 \\
&- (2\cdot 10-1-9)!\binom{10}{9}126u^5v^4 + (2\cdot 10-1-9)!\binom{10}{9}84u^3v^6 \\
&\qquad - (2\cdot 10-1-9)!\binom{10}{9}9uv^8 \\
&+ (2\cdot 10-1-10)!\binom{10}{10}u^{10} - (2\cdot 10-1-10)!\binom{10}{10}45u^8v^2 \\
&\qquad + (2\cdot 10-1-10)!\binom{10}{10}210u^6v^4 \\
&- (2\cdot 10-1-10)!\binom{10}{10}210u^4v^6 + (2\cdot 10-1-10)!\binom{10}{10}45u^2v^8 \\
&\qquad - (2\cdot 10-1-10)!\binom{10}{10}v^{10}
\end{aligned}
\tag{2.2.15}
$$

$$
\begin{aligned}
D(u,v) = &- (2\cdot 10-1-1)!\binom{10}{1}v + (2\cdot 10-1-2)!\binom{10}{2}2uv \\
&- (2\cdot 10-1-3)!\binom{10}{3}3u^2v
\end{aligned}
$$

$$+(2\cdot 10-1-3)!\binom{10}{3}v^3+(2\cdot 10-1-4)!\binom{10}{4}4u^3v$$
$$-(2\cdot 10-1-4)!\binom{10}{4}4uv^3$$
$$-(2\cdot 10-1-5)!\binom{10}{5}5u^4v+(2\cdot 10-1-5)!\binom{10}{5}10u^2v^3$$
$$-(2\cdot 10-1-5)!\binom{10}{5}v^5$$
$$+(2\cdot 10-1-6)!\binom{10}{6}6u^5v-(2\cdot 10-1-6)!\binom{10}{6}20u^3v^3$$
$$+(2\cdot 10-1-6)!\binom{10}{6}6uv^5$$
$$-(2\cdot 10-1-7)!\binom{10}{7}7u^6v+(2\cdot 10-1-7)!\binom{10}{7}35u^4v^3$$
$$-(2\cdot 10-1-7)!\binom{10}{7}21u^2v^5$$
$$+(2\cdot 10-1-7)!\binom{10}{7}v^7+(2\cdot 10-1-8)!\binom{10}{8}8u^7v$$
$$-(2\cdot 10-1-8)!\binom{10}{8}56u^5v^3$$
$$+(2\cdot 10-1-8)!\binom{10}{8}56u^3v^5-(2\cdot 10-1-8)!\binom{10}{8}8uv^7$$
$$-(2\cdot 10-1-9)!\binom{10}{9}9u^8v$$
$$+(2\cdot 10-1-9)!\binom{10}{9}84u^6v^3-(2\cdot 10-1-9)!\binom{10}{9}126u^4v^5$$
$$+(2\cdot 10-1-9)!\binom{10}{9}36u^2v^7$$
$$-(2\cdot 10-1-9)!\binom{10}{9}v^9+(2\cdot 10-1-10)!\binom{10}{10}10u^9v$$
$$-(2\cdot 10-1-10)!\binom{10}{10}120u^7v^3$$
$$+(2\cdot 10-1-10)!\binom{10}{10}252u^5v^5-(2\cdot 10-1-10)!\binom{10}{10}120u^3v^7$$
$$+(2\cdot 10-1-10)!\binom{10}{10}10uv^9$$

(2.2.16)

Theorem:

The integrator is A – stable.

Proof:

All that is needed from the foregoing (2.2.7) is to show that $F(u,v)\le 0,\ \forall\, u<0$.

Indeed, from the expansions for $A(u,v), B(u,v), C(u,v)$ and $D(u,v)$ we know that for the terms in A and B, the variable r range from 0 to 9 while for those in$C$ and D, the variable r range from 0 to 10.

This implies $C_r^{10}\ge C_r^{9}$

Hence in the differences after squaring, we have

$$F(u,v)=[A^2-C^2]+[B^2-D^2]\le 0,\qquad \forall\, u<0$$

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Stability in Polar Variable

On the same integrator, to get the polar form of the stability function, we set $\bar{h} = u + iv$ where $i = \sqrt{-1}$ and then introduce the transformation $u = Rcos\theta$ and $v = Rsin\theta$ so that $\bar{h} = R(cos\theta + isin\theta)$.

We replace the variables u and v with the polar co-ordinates R and θ and hence

$$S(\bar{h}) = S(u, v) = S(R, \theta)$$

Therefore, for $k = 10$, (1.8) becomes

$$\begin{aligned} S(R,\theta) = [&1 \times 10^{17} + 6 \times 10^{16}R(cos\theta + isin\theta) + 1 \times 10^{16}R^2(cos\theta + isin\theta)^2 \\ &+ 2 \times 10^{15}R^3(cos\theta + isin\theta)^3 + 2 \times 10^{14}R^4(cos\theta + isin\theta)^4 \\ &+ 1 \times 10^{13}R^5(cos\theta + isin\theta)^5 + 5 \times 10^{11}R^6(cos\theta + isin\theta)^6 \\ &+ 2 \times 10^{10}R^7(cos\theta + isin\theta)^7 + 359{,}251{,}200R^8(cos\theta + isin\theta)^8 \\ &+ 3{,}638{,}800R^9(cos\theta + isin\theta)^9]/ \\ &[1 \times 10^{17} - 6 \times 10^{16}R(cos\theta + isin\theta) + 2 \times 10^{16}R^2(cos\theta + isin\theta)^2 \\ &-3 \times 10^{15}R^3(cos\theta + isin\theta)^3 + 3 \times 10^{14}R^4(cos\theta + isin\theta)^4 \\ &-2 \times 10^{13}R^5(cos\theta + isin\theta)^5 + 1 \times 10^{12}R^6(cos\theta + isin\theta)^6 \\ &-6 \times 10^{10}R^7(cos\theta + isin\theta)^7 + 1{,}796{,}256{,}000R^8(cos\theta + isin\theta)^8 \\ &- 36{,}288{,}000R^9(cos\theta + isin\theta)^9 + 362{,}880R^{10}(cos\theta + isin\theta)^{10}] \end{aligned} \quad (2.3.1)$$

Applying De Moivres' theorem, we have

$$\begin{aligned} S(R,\theta) = [&(1 \times 10^{17} + 6 \times 10^{16}Rcos\theta + 1 \times 10^{16}R^2cos2\theta \\ &+ 2 \times 10^{15}R^3cos3\theta + 2 \times 10^{14}R^4cos4\theta + 1 \times 10^{13}R^5cos5\theta \\ &+ 5 \times 10^{11}R^6cos6\theta + 2 \times 10^{10}R^7cos7\theta \\ &+ 359{,}251{,}200R^8cos8\theta + 3{,}638{,}800R^9cos9\theta) \\ &+ i(6 \times 10^{16}Rsin\theta + 1 \times 10^{16}R^2sin2\theta + 2 \times 10^{15}R^3sin3\theta \\ &+ 2 \times 10^{14}R^4sin4\theta + 1 \times 10^{13}R^5sin5\theta + 5 \times 10^{11}R^6sin6\theta \\ &+ 2 \times 10^{10}sin7\theta + 359{,}251{,}200R^8sin8\theta \\ &+ 3{,}638{,}800R^9sin9\theta)]/[(1 \times 10^{17} - 6 \times 10^{16}Rcos\theta \\ &+ 2 \times 10^{16}R^2cos2\theta - 3 \times 10^{15}R^3cos3\theta + 3 \times 10^{14}R^4cos4\theta \\ &- 2 \times 10^{13}R^5cos5\theta + 1 \times 10^{12}R^6cos6\theta - 6 \times 10^{10}R^7cos7\theta \\ &+ 1{,}796{,}256{,}000R^8cos8\theta - 36{,}288{,}000R^9cos9\theta \\ &+ 362{,}880R^{10}cos10\theta) + i(1 \times 10^{17} - 6 \times 10^{16}Rsin\theta \\ &+ 2 \times 10^{16}R^2sin2\theta - 3 \times 10^{15}R^3sin3\theta + 3 \times 10^{14}R^4sin4\theta \\ &- 2 \times 10^{13}R^5sin5\theta + 1 \times 10^{12}R^6sin6\theta - 6 \times 10^{10}R^7sin7\theta \\ &+ 1{,}796{,}256{,}000R^8sin8\theta - 36{,}288{,}000R^9sin9\theta \end{aligned}$$

$$+ 362{,}880R^{10}sin10\theta)] \qquad (2.3.2)$$

That is, $|S(R,\theta)|^2 = \dfrac{|\phi(R,\theta)|^2}{|\psi(R,\theta)|^2}$, *where*

$$|\phi(R,\theta)|^2 = 1\times10^{34} + 3\times10^{33}R^2cos^2\theta + 2\times10^{32}R^4cos^22\theta$$
$$+ 3\times10^{30}R^6cos^23\theta + 3\times10^{28}R^8cos^24\theta$$
$$+ 1\times10^{26}R^{10}cos^25\theta + 3\times10^{23}R^{12}cos^26\theta$$
$$+ 3\times10^{20}R^{14}cos^27\theta + 1\times10^{17}R^{16}cos^28\theta$$
$$+ 1\times10^{13}R^{18}cos^29\theta + 1\times10^{34}Rcos\theta$$
$$+ 3\times10^{33}R^2cos2\theta + 4\times10^{32}R^3cos3\theta$$
$$+ 4\times10^{31}R^4cos4\theta + 3\times10^{30}R^5cos5\theta$$
$$+ 1\times10^{29}R^6cos6\theta + 4\times10^{27}R^7cos7\theta$$
$$+ 9\times10^{25}R^8cos8\theta + 9\times10^{23}R^9cos9\theta$$
$$+ 2\times10^{33}R^3cos\theta cos2\theta + 2\times10^{32}R^4cos\theta cos3\theta$$
$$+ 2\times10^{31}R^5cos\theta cos4\theta + 1\times10^{30}R^6cos\theta cos5\theta$$
$$+ 6\times10^{28}R^7cos\theta cos6\theta + 2\times10^{27}R^8cos\theta cos7\theta$$
$$+ 4\times10^{25}R^9cos\theta cos8\theta + 4\times10^{23}R^{10}cos\theta cos9\theta$$
$$+ 5\times10^{31}R^5cos2\theta cos3\theta + 4\times10^{30}R^6cos2\theta cos4\theta$$
$$+ 3\times10^{29}R^7cos2\theta cos5\theta + 1\times10^{28}R^8cos2\theta cos6\theta$$
$$+ 4\times10^{26}R^9cos2\theta cos7\theta + 9\times10^{24}R^{10}cos2\theta cos8\theta$$
$$+ 9\times10^{22}R^{11}cos2\theta cos9\theta + 6\times10^{29}R^7cos3\theta cos4\theta$$
$$+ 4\times10^{28}R^8cos3\theta cos5\theta + 2\times10^{27}R^9cos3\theta cos6\theta$$
$$+ 6\times10^{25}R^{10}cos3\theta cos7\theta + 1\times10^{24}R^{11}cos3\theta cos8\theta$$
$$+ 1\times10^{22}R^{12}cos3\theta cos9\theta + 4\times10^{27}R^9cos4\theta cos5\theta$$
$$+ 2\times10^{26}R^{10}cos4\theta cos6\theta + 6\times10^{24}R^{11}cos4\theta cos7\theta$$
$$+ 2\times10^{23}R^{12}cos4\theta cos8\theta + 1\times10^{21}R^{13}cos4\theta cos9\theta$$
$$+ 1\times10^{25}R^{11}cos5\theta cos6\theta + 4\times10^{23}R^{12}cos5\theta cos7\theta$$
$$+ 8\times10^{21}R^{13}cos5\theta cos8\theta + 8\times10^{19}R^{14}cos5\theta cos9\theta$$
$$+ 2\times10^{22}R^{13}cos6\theta cos7\theta + 4\times10^{20}R^{14}cos6\theta cos8\theta$$
$$+ 4\times10^{18}R^{15}cos6\theta cos9\theta + 1\times10^{19}R^{15}cos7\theta cos8\theta$$
$$+ 1\times10^{16}R^{16}cos7\theta cos9\theta + 3\times10^{15}R^{17}cos8\theta cos9\theta$$
$$+ 3\times10^{33}R^2sin^2\theta + 2\times10^{32}R^4sin^22\theta$$
$$+ 3\times10^{30}R^6sin^23\theta + 3\times10^{28}R^8sin^24\theta$$
$$+ 1\times10^{26}R^{10}sin^25\theta + 3\times10^{23}R^{12}sin^26\theta$$

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$$
\begin{aligned}
&+3\times10^{20}R^{14}sin^{2}7\theta+1\times10^{17}R^{16}sin^{2}8\theta\\
&+1\times10^{13}R^{18}sin^{2}9\theta+2\times10^{33}R^{3}sin\theta sin2\theta\\
&+2\times10^{32}R^{4}sin\theta sin3\theta+2\times10^{31}R^{5}sin\theta sin4\theta\\
&+1\times10^{30}R^{6}sin\theta sin5\theta+6\times10^{28}R^{7}sin\theta sin6\theta\\
&+2\times10^{27}R^{8}sin\theta sin7\theta+4\times10^{25}R^{9}sin\theta sin8\theta\\
&+4\times10^{23}R^{10}sin\theta sin9\theta+5\times10^{31}R^{5}sin2\theta sin3\theta\\
&+4\times10^{30}R^{6}sin2\theta sin4\theta+3\times10^{29}R^{7}sin2\theta sin5\theta\\
&+1\times10^{28}R^{8}sin2\theta sin6\theta+4\times10^{26}R^{9}sin2\theta sin7\theta\\
&+9\times10^{24}R^{10}sin2\theta sin8\theta+9\times10^{22}R^{11}sin2\theta sin9\theta\\
&+6\times10^{29}R^{7}sin3\theta sin4\theta+4\times10^{28}R^{8}sin3\theta sin5\theta\\
&+2\times10^{27}R^{9}sin3\theta sin6\theta+6\times10^{25}R^{10}sin3\theta sin7\theta\\
&+1\times10^{24}R^{11}sin3\theta sin8\theta+1\times10^{22}R^{12}sin3\theta sin9\theta\\
&+4\times10^{27}R^{9}sin4\theta sin5\theta+2\times10^{26}R^{10}sin4\theta sin6\theta\\
&+6\times10^{24}R^{11}sin4\theta sin7\theta+2\times10^{23}R^{12}sin4\theta sin8\theta\\
&+1\times10^{21}R^{13}sin4\theta sin9\theta+1\times10^{25}R^{11}sin5\theta sin6\theta\\
&+4\times10^{23}R^{12}sin5\theta sin7\theta+8\times10^{21}R^{13}sin5\theta sin8\theta\\
&+8\times10^{19}R^{14}sin5\theta sin9\theta+2\times10^{22}R^{13}sin6\theta sin7\theta\\
&+4\times10^{20}R^{14}sin6\theta sin8\theta+4\times10^{18}R^{15}sin6\theta sin9\theta\\
&+1\times10^{19}R^{15}sin7\theta sin8\theta+1\times10^{16}R^{16}sin7\theta sin9\theta\\
&+3\times10^{15}R^{17}sin8\theta sin9\theta
\end{aligned}
\tag{2.3.3}
$$

and

$$
\begin{aligned}
|\psi(R,\theta)|^{2}=\;&1\times10^{34}+4\times10^{33}R^{2}cos^{2}\theta+3\times10^{32}R^{4}cos^{2}2\theta\\
&+6\times10^{30}R^{6}cos^{2}3\theta+8\times10^{28}R^{8}cos^{2}4\theta\\
&+5\times10^{26}R^{10}cos^{2}5\theta+2\times10^{24}R^{12}cos^{2}6\theta\\
&+3\times10^{21}R^{14}cos^{2}7\theta+3\times10^{18}R^{16}cos^{2}8\theta\\
&+1\times10^{15}R^{18}cos^{2}9\theta+1\times10^{11}R^{20}cos^{2}10\theta\\
&-2\times10^{34}Rcos\theta+4\times10^{33}R^{2}cos2\theta-6\times10^{32}R^{3}cos3\theta\\
&+7\times10^{31}R^{4}cos4\theta-5\times10^{30}R^{5}cos5\theta\\
&+3\times10^{29}R^{6}cos6\theta-1\times10^{28}R^{7}cos7\theta\\
&+4\times10^{26}R^{8}cos8\theta-9\times10^{24}R^{9}cos9\theta\\
&+9\times10^{22}R^{10}cos10\theta-2\times10^{33}R^{3}cos\theta cos2\theta\\
&+3\times10^{32}R^{4}cos\theta cos3\theta-4\times10^{31}R^{5}cos\theta cos4\theta
\end{aligned}
$$

$$+3\times10^{30}R^{6}cos\theta cos5\theta-2\times10^{29}R^{7}cos\theta cos6\theta$$
$$+7\times10^{27}R^{8}cos\theta cos7\theta-2\times10^{26}R^{9}cos\theta cos8\theta$$
$$+5\times10^{24}R^{10}cos\theta cos9\theta-5\times10^{22}R^{11}cos\theta cos10\theta$$
$$-8\times10^{31}R^{5}cos2\theta cos3\theta+9\times10^{30}R^{6}cos2\theta cos4\theta$$
$$-7\times10^{29}R^{7}cos2\theta cos5\theta+4\times10^{28}R^{8}cos2\theta cos6\theta$$
$$-2\times10^{27}R^{9}cos2\theta cos7\theta+6\times10^{25}R^{10}cos2\theta cos8\theta$$
$$-1\times10^{24}R^{11}cos2\theta cos9\theta+1\times10^{22}R^{12}cos2\theta cos10\theta$$
$$-1\times10^{30}R^{7}cos3\theta cos4\theta+1\times10^{29}R^{8}cos3\theta cos5\theta$$
$$-7\times10^{27}R^{9}cos3\theta cos6\theta+3\times10^{26}R^{10}cos3\theta cos7\theta$$
$$-9\times10^{24}R^{11}cos3\theta cos8\theta+2\times10^{23}R^{12}cos3\theta cos9\theta$$
$$-2\times10^{21}R^{13}cos3\theta cos10\theta-1\times10^{28}R^{9}cos4\theta cos5\theta$$
$$+7\times10^{26}R^{10}cos4\theta cos6\theta-3\times10^{25}R^{11}cos4\theta cos7\theta$$
$$+10\times10^{23}R^{12}cos4\theta cos8\theta-2\times10^{22}R^{13}cos4\theta cos9\theta$$
$$+2\times10^{20}R^{14}cos4\theta cos10\theta-6\times10^{25}R^{11}cos5\theta cos6\theta$$
$$+3\times10^{24}R^{12}cos5\theta cos7\theta-8\times10^{22}R^{13}cos5\theta cos8\theta$$
$$+2\times10^{21}R^{14}cos5\theta cos9\theta-2\times10^{19}R^{15}cos5\theta cos10\theta$$
$$-2\times10^{23}R^{13}cos6\theta cos7\theta+5\times10^{21}R^{14}cos6\theta cos8\theta$$
$$-10\times10^{19}R^{15}cos6\theta cos9\theta+10\times10^{17}R^{16}cos6\theta cos10\theta$$
$$-2\times10^{20}R^{15}cos7\theta cos8\theta+4\times10^{18}R^{16}cos7\theta cos9\theta$$
$$-4\times10^{16}R^{17}cos7\theta cos10\theta-1\times10^{17}R^{17}cos8\theta cos9\theta$$
$$+1\times10^{15}R^{18}cos8\theta cos10\theta-3\times10^{13}R^{19}cos9\theta cos10\theta$$
$$+4\times10^{33}R^{2}sin^{2}\theta+3\times10^{32}R^{4}sin^{2}2\theta$$
$$+6\times10^{30}R^{6}sin^{2}3\theta+8\times10^{28}R^{8}sin^{2}4\theta$$
$$+5\times10^{26}R^{10}sin^{2}5\theta+2\times10^{24}R^{12}sin^{2}6\theta$$
$$+3\times10^{21}R^{14}sin^{2}7\theta+3\times10^{18}R^{16}sin^{2}8\theta$$
$$+1\times10^{15}R^{18}sin^{2}9\theta+1\times10^{11}R^{20}sin^{2}10\theta$$
$$-2\times10^{33}R^{3}sin\theta sin2\theta+3\times10^{32}R^{4}sin\theta sin3\theta$$
$$-4\times10^{31}R^{5}sin\theta sin4\theta+3\times10^{30}R^{6}sin\theta sin5\theta$$
$$-2\times10^{29}R^{7}sin\theta sin6\theta+7\times10^{27}R^{8}sin\theta sin7\theta$$
$$-2\times10^{26}R^{9}sin\theta sin8\theta+5\times10^{24}R^{10}sin\theta sin9\theta$$
$$-5\times10^{22}R^{11}sin\theta sin10\theta-8\times10^{31}R^{5}sin2\theta sin3\theta$$
$$+9\times10^{30}R^{6}sin2\theta sin4\theta-7\times10^{29}R^{7}sin2\theta sin5\theta$$
$$+4\times10^{28}R^{8}sin2\theta sin6\theta-2\times10^{27}R^{9}sin2\theta sin7\theta$$
$$+6\times10^{25}R^{10}sin2\theta sin8\theta-1\times10^{24}R^{11}sin2\theta sin9\theta$$

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$$+1\times10^{22}R^{12}sin2\theta sin10\theta-1\times10^{30}R^{7}sin3\theta sin4\theta$$
$$+1\times10^{29}R^{8}sin3\theta sin5\theta-7\times10^{27}R^{9}sin3\theta sin6\theta$$
$$+3\times10^{26}R^{10}sin3\theta sin7\theta-9\times10^{24}R^{11}sin3\theta sin8\theta$$
$$+2\times10^{23}R^{12}sin3\theta sin9\theta-2\times10^{21}R^{13}sin3\theta sin10\theta$$
$$-1\times10^{28}R^{9}sin4\theta sin5\theta+7\times10^{26}R^{10}sin4\theta sin6\theta$$
$$-3\times10^{25}R^{11}sin4\theta sin7\theta+10\times10^{23}R^{12}sin4\theta sin8\theta$$
$$-2\times10^{22}R^{13}sin4\theta sin9\theta+2\times10^{20}R^{14}sin4\theta sin10\theta$$
$$-6\times10^{25}R^{11}sin5\theta sin6\theta+3\times10^{24}R^{12}sin5\theta sin7\theta$$
$$-8\times10^{22}R^{13}sin5\theta sin8\theta+2\times10^{21}R^{14}sin5\theta sin9\theta$$
$$-2\times10^{19}R^{15}sin5\theta sin10\theta-2\times10^{23}R^{13}sin6\theta sin7\theta$$
$$+5\times10^{21}R^{14}sin6\theta sin8\theta-10\times10^{19}R^{15}sin6\theta sin9\theta$$
$$+10\times10^{17}R^{16}sin6\theta sin10\theta-2\times10^{20}R^{15}sin7\theta sin8\theta$$
$$+4\times10^{18}R^{16}sin7\theta sin9\theta-4\times10^{16}R^{17}sin7\theta sin10\theta$$
$$-1\times10^{17}R^{17}sin8\theta sin9\theta+1\times10^{15}R^{18}sin8\theta sin10\theta$$
$$-3\times10^{13}R^{19}sin9\theta sin10\theta$$

(2.3.4)

Simplifying further using the trigonometric identity which says that $cosA\pm B=cosAcoB\pm sinAsinB$.Therefore, we set $F(R,\theta)=|\psi(R,\theta)|^2-|\phi(R,\theta)|^2\geq0$.

That is,

$$F(R,\theta)=R[(-2\cdot97\times10^{34}cos\theta)$$
$$+R(8\times10^{32}+8\times10^{32}cos2\theta)$$
$$+R^2(-1\times10^{33}cos3\theta-4\times10^{33}cos\theta)$$
$$+R^3(9\times10^{31}+3\times10^{31}cos4\theta+1\times10^{32}cos2\theta)$$
$$+R^4(-8\times10^{30}cos5\theta-5\times10^{31}cos3\theta-1\times10^{32}cos\theta)$$
$$+R^5(3\times10^{30}+2\times10^{29}cos6\theta+2\times10^{30}cos4\theta+5\times10^{30}cos2\theta)$$
$$+R^6(-2\times10^{28}cos7\theta-8\times10^{29}cos5\theta-10\times10^{29}cos3\theta-2\times10^{30}cos\theta)$$
$$+R^7(5\times10^{28}+4\times10^{26}cos8\theta+5\times10^{27}cos6\theta+3\times10^{28}cos4\theta+7\times10^{28}cos2\theta)$$
$$+R^8(-10\times10^{24}cos9\theta-3\times10^{26}cos7\theta-12\times10^{27}cos5\theta-8\times10^{27}cos3\theta-2\times10^{28}cos\theta)$$
$$+R^9(4\times10^{26}+9\times10^{22}cos10\theta+4\times10^{24}cos8\theta+5\times10^{25}cos6\theta+2\times10^{26}cos4\theta+5\times10^{26}cos2\theta)$$
$$+R^{10}(-5\times10^{22}cos9\theta-1\times10^{24}cos7\theta-1\times10^{25}cos5\theta-4\times10^{25}cos3\theta-7\times10^{25}cos\theta)$$

$$+ R^{11}(1 \times 10^{24} + 1 \times 10^{22}cos8\theta + 2 \times 10^{23}cos6\theta + 9 \times 10^{23}cos4\theta + 2 \times 10^{24}cos2\theta)$$
$$+ R^{12}(- 2 \times 10^{21}cos7\theta - 2 \times 10^{22}cos5\theta - 8 \times 10^{23}cos3\theta - 2 \times 10^{23}cos\theta)$$
$$+ R^{13}(3 \times 10^{21} + 2 \times 10^{20}cos6\theta + 2 \times 10^{21}cos4\theta + 4 \times 10^{21}cos2\theta)$$
$$+ R^{14}(-2 \times 10^{19}cos5\theta - 10 \times 10^{19}cos3\theta - 2 \times 10^{20}cos\theta)$$
$$+ R^{15}(3 \times 10^{18} + 10 \times 10^{17}cos4\theta + 4 \times 10^{18}cos2\theta)$$
$$+ R^{16}(- 4 \times 10^{16}cos3\theta - 1 \times 10^{17}cos\theta)$$
$$+ R^{17}(1 \times 10^{15} + 1 \times 10^{15}cos2\theta)$$
$$+ R^{18}(- 3 \times 10^{13}cos\theta) + R^{19}(1 \times 10^{11})] \geq 0$$

(2.3.5)

The region of absolute stability is the polar curve $F(R,\theta) = 0$, where$u = Rcos\theta$and$v = Rsin\theta$.

THE JORDAN CURVE

The Jordan curve is the equation governing the region of absolute stability and the region of instability.

To construct the Jordan curve, equation (2.3.5) becomes handy. We take various degrees of θ from 0^o to 360^o and get a corresponding value of R.

These values are now plotted to get the Jordan curve.

The table below gives the value of θ and the corresponding R.

Table 1. Analysis of the Jordan curve, $F(R, \emptyset)$

θ in Degree	*Approximate Value for the Curve $F(R,\theta)$*	*θ in Degree*	*Approximate Value for the Curve $F(R,\theta)$*
0	99.6	190	-98.8
10	98.8	200	-97.9
20	97.9	210	-96.8
30	96.8	220	-94.8
40	94.8	230	-90.8
50	90.8	240	-88.2
60	88.2	250	-84.1
70	84.1	260	-81.6
80	81.6	270	0
90	0	280	81.6
100	-81.6	290	84.1
110	-84.1	300	88.2

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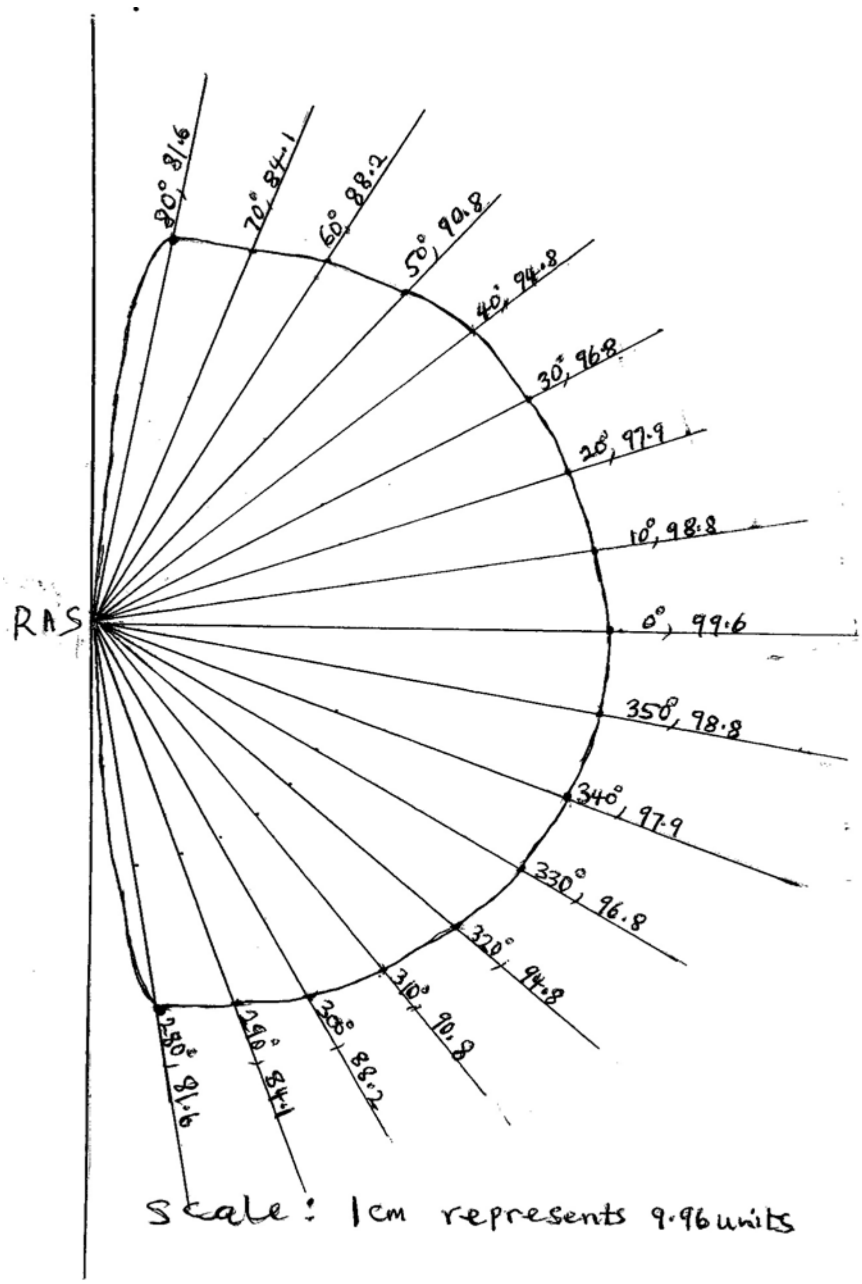
120	-88.2	310	90.8
130	-90.8	320	94.8
140	-94.8	330	96.8
150	-96.8	340	97.9
160	-97.9	350	98.8
170	-98.8	360	99.6
180	-99.6		



APPLICATIONS

Problem, Ademiluyi (2005)

$y' = 1 + y^2, \;\; y(0) = 1, \;\; 0 \le x \le 1$

Inverse Runge-Kutta Method of Order 2

N	x_n	$y(x_n)$	y_n	E_n	*Rational Integrator K=10*	*Error*
1	0.1000	1.223048884	1.234567905	-0.1152D-01	1.2230D+00	1.6017D-06
2	0.2000	1.508497657	1.538880977	-0.3038D-01	1.5085D+00	1.0549D-12
3	0.3000	1.895765178	1.958484913	-0.6272D-01	1.8958D+00	-1.4643D-11
4	0.4000	2.464962799	2.589525967	-0.1246D+00	2.4650D+00	-1.0355D-10
5	0.5000	3.408223442	3.674915625	-0.2667D+00	3.4082D+00	-7.6033D-10
6	0.6000	5.331855925	6.056638867	-0.7248D+00	5.3319D+00	-8.1347D-09
7	0.7000	11.681380355	15.994363155	-0.4313D+01	1.0621D+01	1.0604D+00
8	0.8000	-68.479332859	-26.420600456	-0.4206D+02	1.0608D+02	-1.7456D+02
9	0.9000	-8.687622253	-7.247147091	-0.1440D+01	-2.5230D+07	2.5230D+07

It can be observed from the above result that our integrator performed better than that of Ademiluyi (2005).

Problem, Sunday and Odekunle (2012)

$y' = 4x - 2xy, \;\; y(0) = 3, \quad h = 0.1$

X	*Numerical Solution*	*Exact Solution*	*Error*	*Rational Integrator K = 10*	*Error*
0.100	2.97044683	2.99004984	0.01960301	2.9704D+00	1.9604D-02
0.200	2.92311978	2.96078944	0.03766966	2.9417D+00	1.9062D-02
0.300	2.86071396	2.91393113	0.05321717	2.8954D+00	1.8521D-02
0.400	2.78663735	2.85214376	0.06550741	2.8327D+00	1.9475D-02
0.500	2.70469856	2.77880073	0.07410216	2.7528D+00	2.6002D-02
0.600	2.61879468	2.69767618	0.07888150	2.6521D+00	4.5547D-02
0.700	2.53260279	2.61262655	0.08002377	2.5259D+00	8.6740D-02
0.800	2.44933867	2.52729249	0.07795382	2.3733D+00	1.5402D-01
0.900	2.37158442	2.44485807	0.07327366	2.2012D+00	2.4369D-01

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It can be observed from the above result that our integrator performed better than that of Sunday and Odekunle (2012).

Problem, Agbeboh and Aashikpelokhai (2007)

$y' = 1 + y^2, \;\; y(0) = 1, \;\; 0 \leq x \leq 1$

XN	*YN*	*TSOL*	*ERROR*	*RATIONAL INTEGRATOR K=10*	*ERROR*
.1D+00	1.2230D+00	0.1223048880450D+01	4.8079D-12	1.2230D+00	1.6017D-06
.2D+00	1.5085D+00	0.1508497647121D+01	4.5104D-12	1.5085D+00	1.0549D-12
.3D+00	1.8958D+00	0.1895765122854D+01	6.6493D-12	1.8958D+00	-1.4643D-11
.4D+00	2.4565D+00	0.2464962756723D+01	8.5126D-03	2.4650D+00	-1.0355D-10
.5D+00	3.3666D+00	0.3408223442336D+01	4.1610D-02	3.4082D+00	-7.6033D-10
.6D+00	5.1379D+00	0.5331855223459D+01	1.9399D-01	5.3319D+00	-8.1347D-09
.7D+00	1.0328D+01	0.1168137380031D+2	1.3535D+00	1.0621D+01	1.0604D+00
.8D+00	8.6318D+00	-.6847966834558D+02	-1.5480D+02	1.0608D+02	-1.7456D+02
.9D+00	4.0934D+06	-.8687629546482D+01	-4.0934D+06	-2.5230D+07	2.5230D+07

From the above result, it can be seen that our integrator compares favourably in terms of accuracy with that of Agbeboh and Aashikpelokhai.

CONCLUSION

In this paper, detailed analysis of the coefficient parameters of the $u - v$ form and the polar form of the stability function were carried out. With these parameters, we were able to determine the region of absolute stability of the order 19 rational integrator.

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