A BRIEF INTRODUCTION OF FRACTIONAL FOURIER TRANSFORMS (FRFT)

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ABSTRACT

Fractional Fourier Transforms (FRFT) is introduced in this study. The basis of the FRFT with nostationary behaviors is a chirp-like function related to sinusoid basis of Fourier Transforms. Therefore, the FRFT could be considered as the Chirp-like Transforms.

Keywords: Fractional Fourier Transforms (FRFT), Basis, Nostationary Behaviors, Chirp-like function, Sinusoid Basis of Fourier Transforms

INTRODUCTION

The fractional Fourier transform (FRFT) is a family of linear transformations to generalize the Fourier transforms (FT) (Ozaktas et al 2001). It can be thought of as the Fourier transform with the order. The order can not be an integer. FRFT can transform a function to any intermediate domain between time and frequency. FRFT can be used in the field of digital signal processing for seismology, de-noising, filter design and signal analysis to pattern recognition etc. The FRFT can be used to fractional convolution, correlation, and other operations and is generalized for the linear canonical transformation (Ozaktas et al 2001). However, it was not widely recognized in signal processing until 1993. FRFT is a tool to extend to Fractional Fourier domain. The basis of FRFT is the chirp-like function. Discrete Fourier transform is shifted by a fractional amount in frequency space and evaluating at a fractional set of different frequencies. FRFT transforms is evaluated efficiently by Fast Fourier transform algorithm (Oppenheim 1999). FRFT has fallen out in the technical applications. In this study, FRFT is described with mathematical analysis and is related to Fourier transform through the chirp-like function.

THEORY OF FRACTIONAL FOURIER TRANSFORMS (FRFT)

For a one-dimensional time function $f(t)$, Its FRFT is $\{F^a f(t)\}(x)$ represented by operator F^a. The parameter a is called the order of ($0<|a|<2$) of FRFT as follows (Ozaktas et al., 1994; Ozaktas et al., 1996);

$$F^a[f(t)]=\{F^a f(t)\}(x)=F^a(x)=\int_{-\infty}^{\infty} B_a(x,t)\,f(t)\,dt \qquad (1)$$

$B_a(x,t)=A_\phi e^{[i\pi(x^2\cot\phi-2xt\csc\phi+t^2\cot\phi)]}$, is the kernel of this integrand.

$A_\phi=\dfrac{e^{[-i\pi\frac{\mathrm{sgn}(\sin\phi)}{4}+i\frac{\phi}{2}]}}{\sqrt{|\sin\phi|}}$, where $\phi=\dfrac{a\pi}{2}$, and i is the imaginary unit.

The parameter x is dimensionless (Ozaktas et al., 2001). The parameter ϕ is the phase angle. The parameter a includes the meaning of the phase because of $\phi = \frac{a\pi}{2}$. If the order a is one, then it back to the Fourier Transforms. If the order a is 0, then the signals have no transformation.

INVERSE FRACTIONAL FOURIER TRANSFORMS (IFRFT)

The inverse Fractional Fourier Transforms (IFRFT) is built using the order of $-a$ in Formula (1).

$$f(t) = \int_{-\infty}^{\infty} B_a(x,t) F^a(x) dx \qquad (2)$$

$B_a(x,t) = A_\phi e^{[i\pi(-x^2 \cot\phi + 2xt\csc\phi - t^2\cot\phi)]}$, is the kernel of this integrand. (3)

$$A_\phi = \frac{e^{[i\pi\frac{\mathrm{sgn}(\sin\phi)}{4} - i\frac{\phi}{2}]}}{\sqrt{|\sin\phi|}}$$, where $\phi = \frac{a\pi}{2}$.

Therefore Formulas (1) and (2) is a FRFT pair.

THE BASES OF FRACTIONAL FOURIER TRANSFORMS AND FOURIER TRANSFORMS

Fourier Transforms (FT)

If the a is one, then Formula (1) becomes the Fourier Transforms. The bases are the different frequency sinusoids with the arguments $\pi(2xt)$, here x becomes the frequency.

Fractional Fourier Transforms (FRFT)

Fro the FRFT of different ϕ, the bases are the sinusoids that include the argument $\pi(-x^2\cot\phi + 2xt\csc\phi - t^2\cot\phi)$ with different x values The parameter x decides the behavior of the FRFT base for a certain order according to $\phi = \frac{a\pi}{2}$, therefore the concept of the Fractional Fourier Transforms likes the Fourier Transforms. The both can think about the transforms with different defined bases.

DISCUSSION

As shown the FRFT had the similar meaning as the Fourier Transforms. The arbitrary shape waveform was combined from the different bases, which have the arguments with nostationary behaviors. The nostationary behavior is defined; the frequency changes as the time changes for a FRFT basis with a certain order. The FRFT basis could be thought as the chirp-like function from Formula (3) shown in Figures 1 to 25 as the example (Note: the basis is only shown during the time period -5sec to 5 sec and the sampling rate of t is 100Hz). Therefore the FRFT could be considered as the Chirp-like Transforms.

CONCLUSION

Some bases of FRFT have been shown in figures 1 to 25; they were chirp-like function. Therefore the FRFT could be considered as the Chirp-like Transforms.

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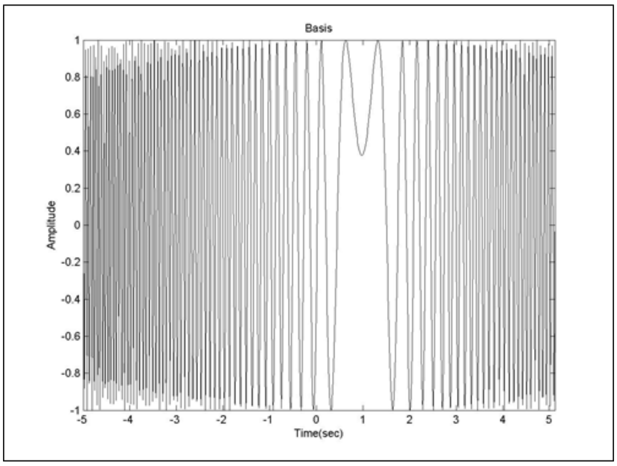
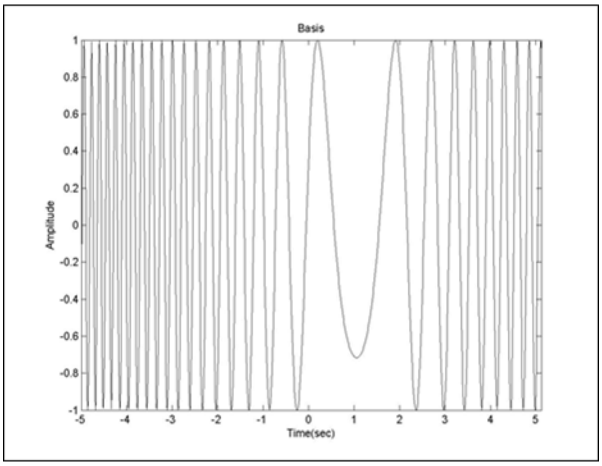


Fig.1: FRFT basis with the order = 0.2 and x =1



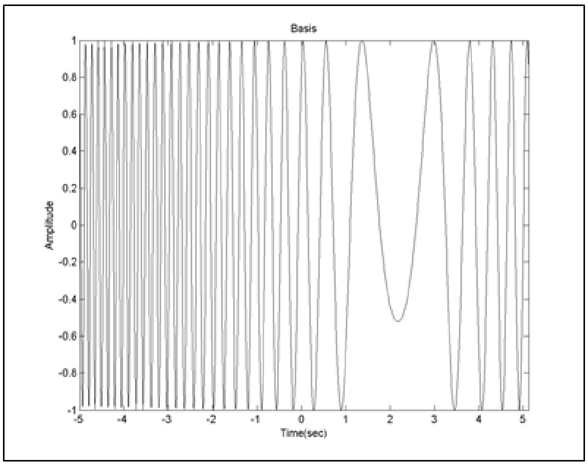
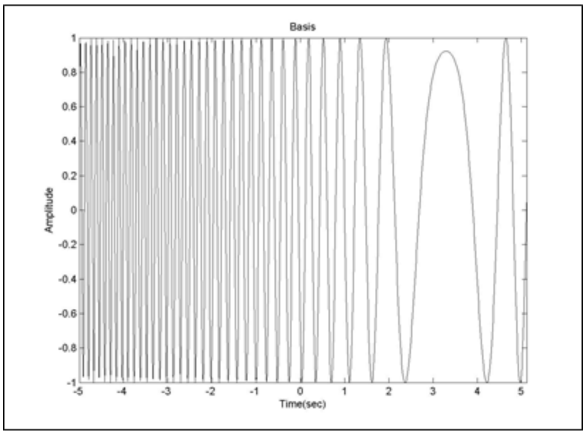


Fig.2: FRFT basis with the orde r= 0.5 and x =1

Fig.3: FRFT basis with the order=0.5 and x =2



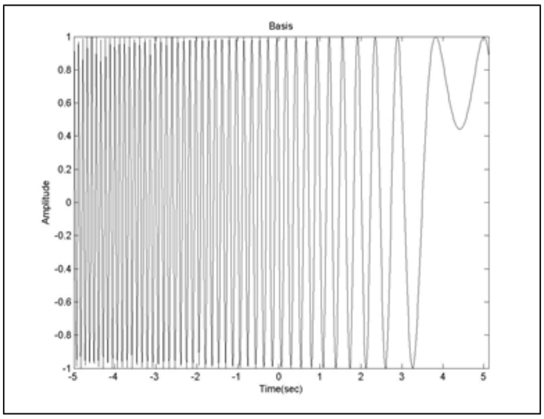


Fig.4: FRFT basis with the order=0.5 and x =3

Fig.5: FRFT basis with the order=0.5 and x =4

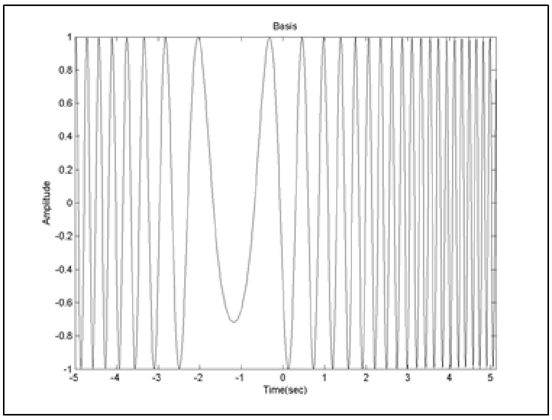


Fig.6: FRFT basis with the order=0.5 and x = -1

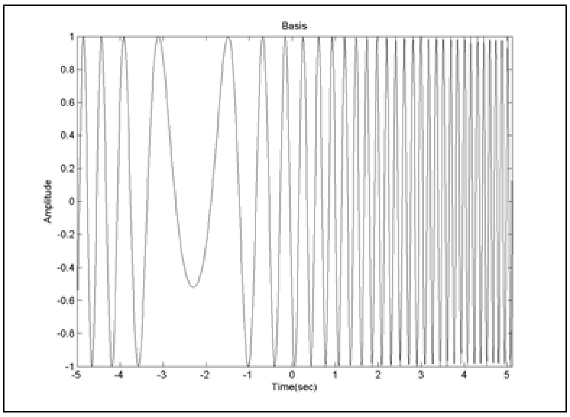


Fig.7: FRFT basis with the order=0.5 and x = -2

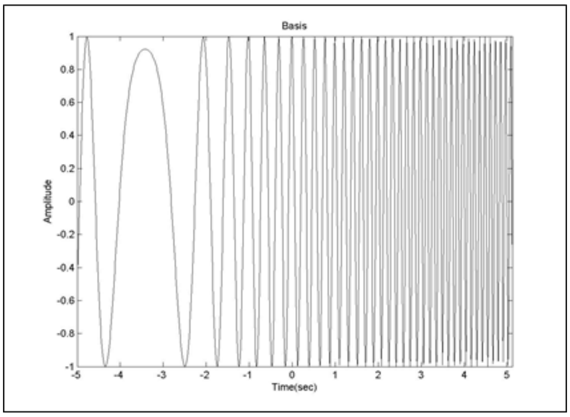


Fig.8: FRFT basis with the order=0.5 and x = -3

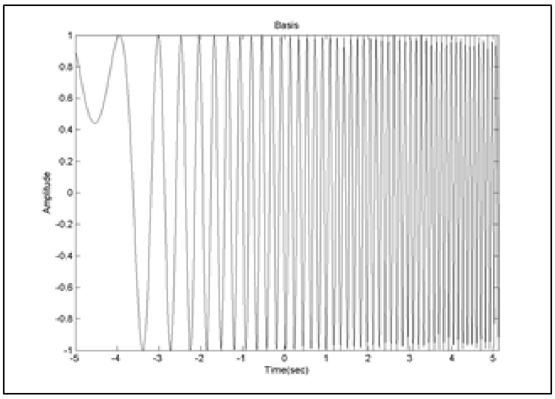


Fig.9: FRFT basis with the order=0.5 and x = -4

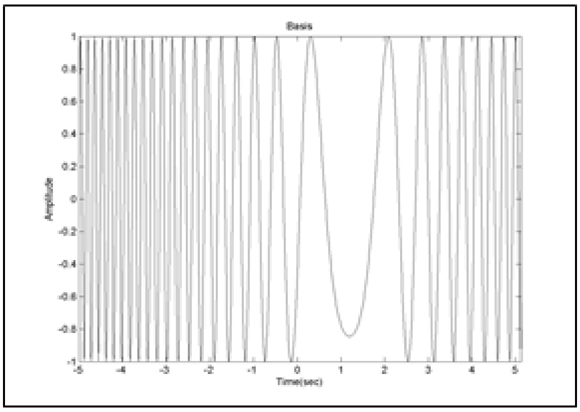


Fig.10: FRFT basis with the order=1.5 and x =1

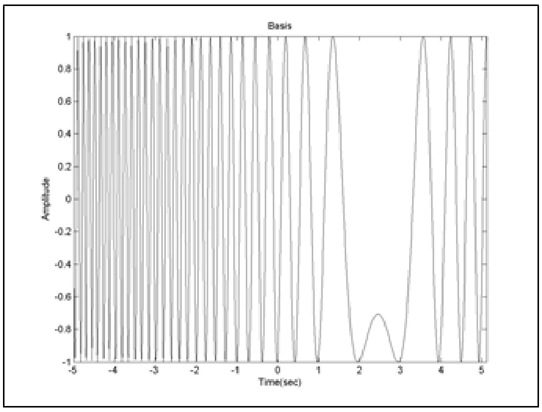


Fig.11: FRFT basis with the order=1.5 and x =2

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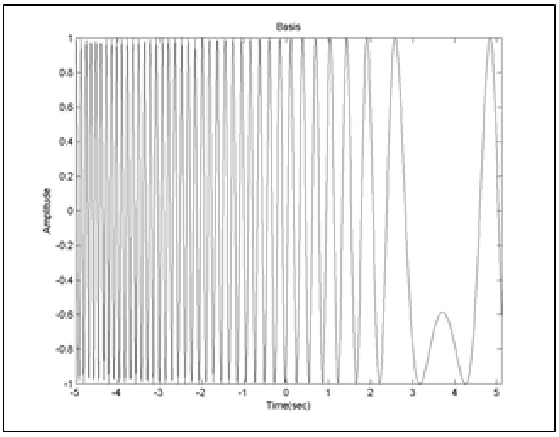

Fig.12: FRFT basis with the order=1.5 and *x* =3

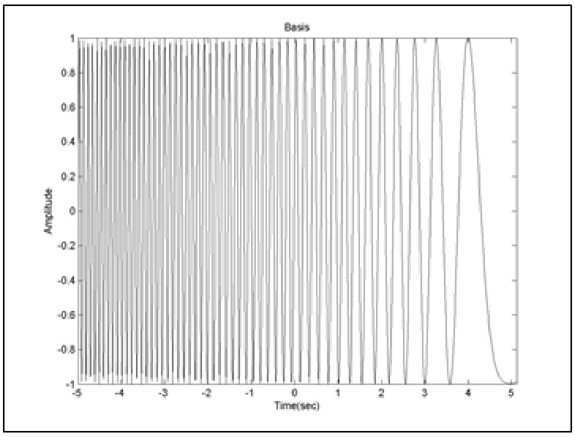

Fig.13: FRFT basis with the order=1.5 and *x* =4

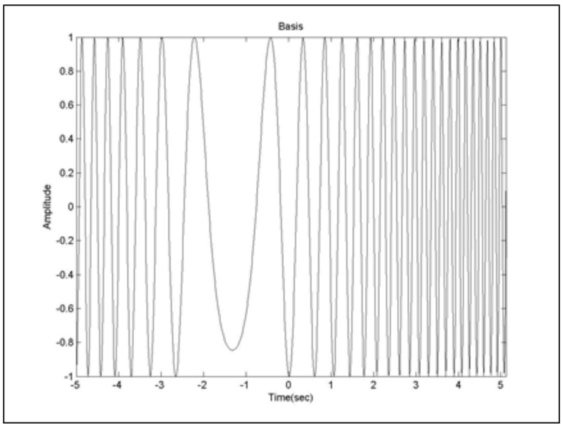

Fig.14: FRFT basis with the order=1.5 and *x* = -1

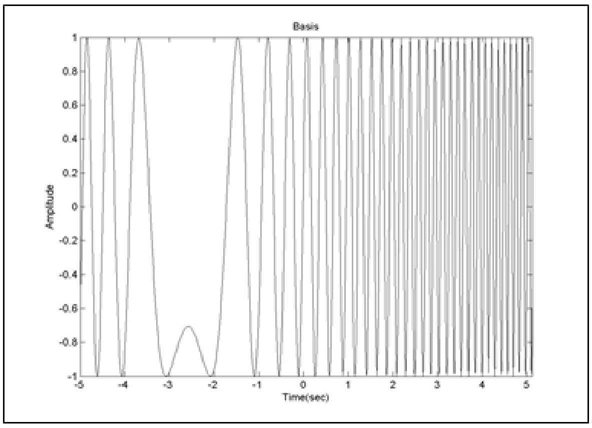

Fig.15: FRFT basis with the order=1.5 and *x* = -2

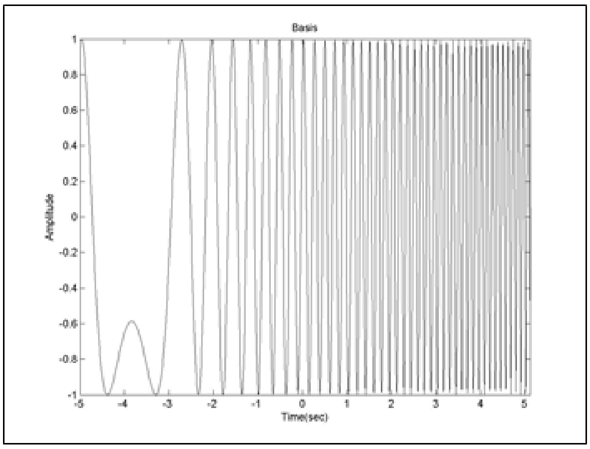

Fig.16: FRFT basis with the order=1.5 and *x* = -3

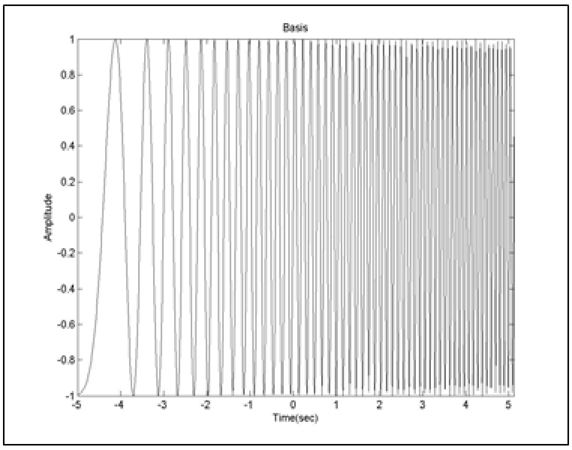

Fig.17: FRFT basis with the order=1.5 and *x* = -4

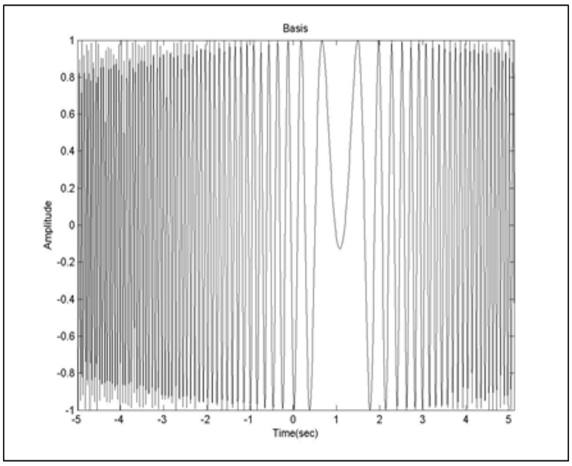

Fig.18: FRFT basis with the order=1.8 and *x* =1

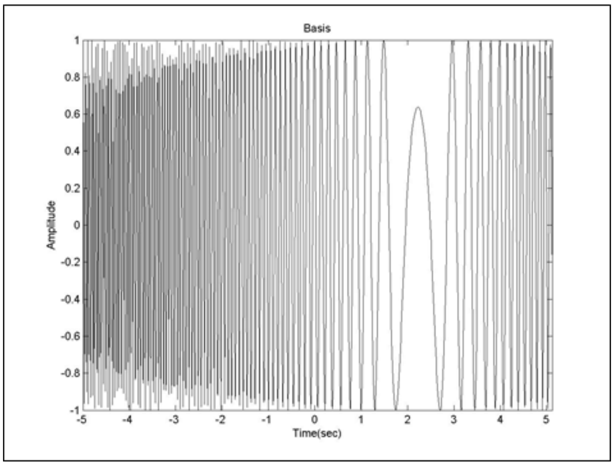

Fig.19: FRFT basis with the order=1.8 and *x* =2

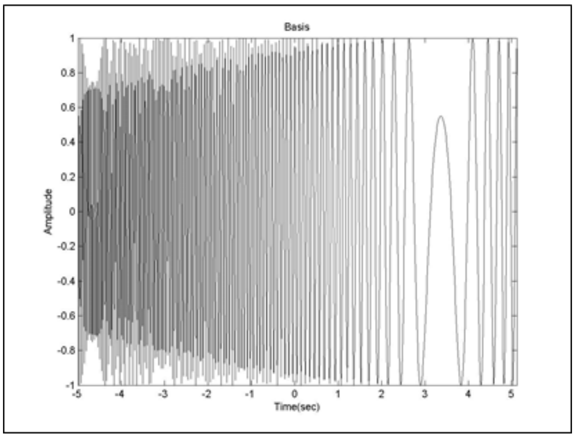

Fig. 20: FRFT basis with the order=1.8 and *x* =3

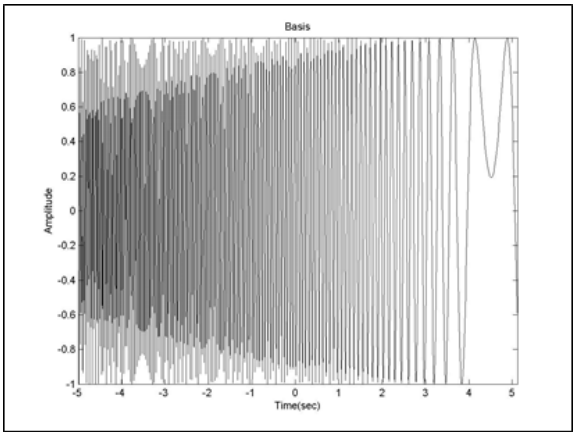

Fig. 21: FRFT basis with the order=1.8 and *x* =4

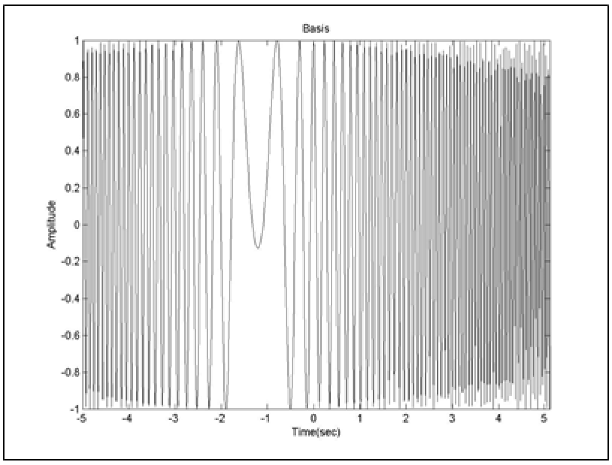

Fig. 22: FRFT basis with the order=1.8 and *x* = -1

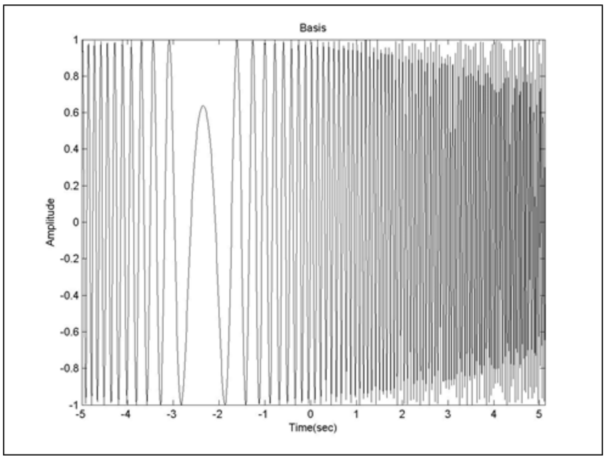

Fig.23: FRFT basis with the order=1.8 and *x* = -2

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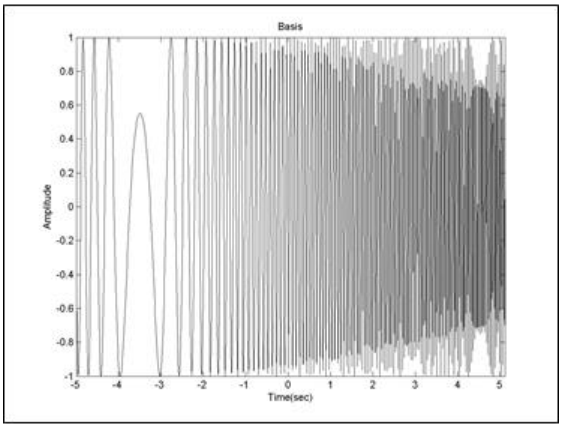

Fig. 24: FRFT basis with the order=1.8 and *x* = -3

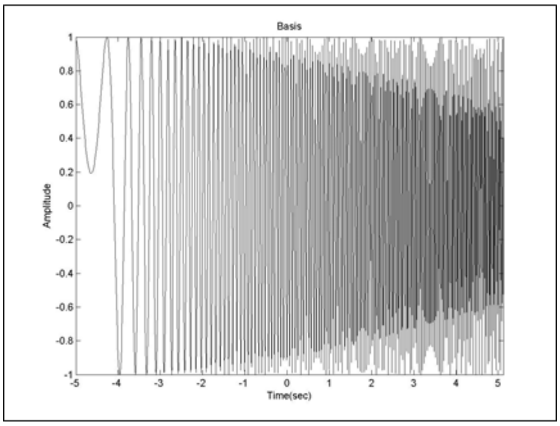

Fig. 25: FRFT basis with the order=1.8 and *x* =-4

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