# Markov 0-Simple Semigroup

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## ABSTRACT

*We constructed Markov 0-simple semi groupusing the notion of Markov semi group and Determine when it is Markov and strongly Markov. Using the constructed Markov 0-simple semigroup and obtaining its properties we concluded that the direct product of two finitely generated markov 0-simple semigroup is Markov. AMS Classification No: 20M05*

**Keywords**: Markov 0-simple semigroup, Regular language, Markov group, Finite state automaton, Markov grammar and Context free grammar.

## INTRODUCTION

The notion of Markov semigroup came as a result of the attempt to generalize the already studied Markov group. The generalization of Markov group to Semigroup was achieved by Cain and Maltcev [2]. A group is Markov if it recognizes a prefix-closed regular language of unique representatives with respect to some generating set and strongly Markov if it recognizes a language of unique minimal-length representatives over every generating set [2]. By this definition of Markov group, they were able to assert when a semigroup is Markov and strongly Markov. A semigroup is Markov if it admits a regular language that does not contain the empty word, is +-prefix-closed and contains a unique representation with respect to some generating set and strongly Markov if there exists a robust semigroup Markov language such that, $|w| = \lambda_A(\overline{w})$[2]. The necessary and sufficient conditions for the direct product of two finitely generated semigroups to be finitely generated [9] was built upon by [3] and as a matter of fact they assert that if $S$ and $T$ are automatic semigroups then, $S \times T$ is automatic if and only if $S \times T$ is finitely generated.

Klimann [8] gave a pictorial illustration of how semigroup can be generated using an automaton $\mathcal{A}$. In his structure, he used the Mealy automata to generate semigroup. By this automaton given by Klimann [8], we were able to deduce that it is actually a semigroup since the properties of semigroup are satisfied. In light of this semigroup automaton we deduce also that a semigroup automaton is an automatic structure that is close and associative by concatenation. In section 2, the definition, terms and concept important to this study are given. Section 3 considers the construction of 0-free semigroup, section 4 deals with Markov 0-simple semigroupand its properties are obtained.And Section 5 deals with the construction of a direct product $(S \times T)$ that is Markov 0-simple semigroup.

## PRELIMINARIES

In this section, the basic concepts needed in this study will be presented below. We follow [1], [2], [3], [4] in our notations. Readers that are not familiar with some of the definitions and terminologies are referred to [2], [3], [4] [5] [6] and [11].

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## Definition of Terms

### *Semigroup*

(i) Let S be a non-empty set with an associative binary operation $*$ defined on it, then S is called a Semigroup.

(ii) Let $S$ be a semigroup without zero, then S is called Simple if it has no proper ideal. That is, if $I$ is an ideal of S, then $I = S$.

(iii) Let $S$ be a semigroup with zero, then $S$ is called 0-simple if, $\{0\}$ and $S$ are the only ideals and $S^2 \neq \{0\}$.

(iv) Let A be a non-empty set and $A^+$ be a set of all finite non-empty words $a_1a_2 \ldots a_m$ in the alphabet A. A binary operation is defined on $A^+$ by juxtaposition:

$$(a_1a_2 \ldots a_m)(b_1b_2 \ldots b_n) = a_1a_2 \ldots a_m b_1b_2 \ldots b_n.$$

With respect to this operation, $A^+$ is a semigroup called the **free semigroup**on A. let us adjoin $\{0\}$ to $A^+$, that is, $A^+ \cup \{0\}$ and make the definition below on it.

We define $a.b = \begin{cases} ab & if\ 0 \notin A^+ \\ 0 & if\ 0 \in A^+ \end{cases}$
where $a = a_1, a_2, \ldots, a_m$and $b = b_1, b_2, \ldots, b_n$. This definition makes$A^+ \cup \{0\}$ 0-free semigroup on $A$ denoted by $A^\#$ and we think of 0 as a word.Also, if we adjoin an identity 1 to $A^+$we obtain $A^*$ called the free monoid, that is, $A^* = A^+ \cup \{1\}$.

Note that,$A^\# = A^+ \cup \{0\}$if $A^+$has nozero elementand$A^+ = A^\#$, if$A^+$ has a zero element. That is,

$$A^\# = \begin{cases} A^+ \cup \{0\} & if\ 0 \notin A^+ \\ A^+ & if\ 0 \in A^+ \end{cases}$$

(v) Let $w = a_1 \ldots a_n (a_i \in A)$ be a non-empty word in A, then the **length of w** denoted by $|\mathbf{w}|$ is the sum of all the elements in w. That is,

$$|w| = \sum_{\sigma_1 \in A} |w|_{\sigma_1}$$

where the length of 0 denoted by $|0| = 1$

(vi) Let$S$ be a semigroup. An element $s \in S$is said to be ***decomposable***if there exist elements $s_1, s_2 \in S$ such that $s = s_1s_2$. Thus the set of all decomposable elements of $S$ is

$$S^2 = SS = \{s_1s_2 : s_1, s_2 \in S\}$$

(vii) An element is ***indecomposable***if it is not decomposable. The set of all indecomposable elements is $S \backslash S^2$. It is clear that this set is contained in every generating set for $S$.

### *Direct Products*

The **direct product** $A \times B$of two semigroups $A$and Bis defined by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, b_1b_2)\ (a_i \in A, b_i \in B).$$

For example, Let A= $(\mathbb{N}, +)$and B= $(\mathbb{N}, \cdot)$. Then in the direct product $A \times B$we have

$(n, r) \cdot (m, s) = (n + m, r.s)(\forall n, m, r, s \in \mathbb{N})$.

The direct product is a convenient way of combining two semigroup operations. The new semigroup $A \times B$inherits the properties of both $A$and $B$.

However, the direct product operation is associative on semigroups; that is, for all $s_1, s_2, \ldots, s_n \in S$ $and$ $t_1, t_2, \ldots, t_n \in T$ we have

$$(s_1, t_1).(s_2, t_2).(s_3, t_3) = [(s_1, t_1).(s_2, t_2)].(s_3, t_3)$$
$$= (s_1 s_2, t_1 t_2).(s_3, t_3)$$
$$= (s_1 s_2 s_3, t_1 t_2 t_3)$$
$$= (s_1, t_1)(s_2 s_3, t_2 t_3)$$

$= (s_1, t_1)[(s_2, t_2).(s_3, t_3)]$.

The **direct product** $A^{\#} \times B^{\#}$of two 0-free semigroup $A^{\#}$and $B^{\#}$is defined by

$$(a_1 a_2 \ldots a_n, \quad b_1 b_2 \ldots b_n) \cup \{0\} = \{[(a_1, b_1)(a_2, b_2) \ldots (a_n, b_n)] \cup \{0\}\} \in A^{\#} \times B^{\#} \forall (a_i \in A^+, b_i \in B^+).$$

Note that, 0 must be in isolation from both $A^+$and $B^+$. Also, the direct product operation is associative for 0-free semigroup and can be illustrated as above respectively.

***Languages***

(i) Let A be a finite alphabet, a **language L** on the set A is a subset of $A^*$. In other words, a language on Ais a (finite or infinite) set of strings of A.

Let A be a finite alphabet, a **language$L^{\#}$** on the set A is a subset of $A^{\#}$. In other words, a language on Ais a (finite or infinite) set of strings of A. Note that $L^{\#} = L^+ \cup \{0\} = L^1 \cup L^2 \cup \ldots \cup \{0\} \forall L^1, L^2, \ldots, L^n \in L^+$

*Regular Languages*

Let $A$be a non-empty finite set. A language $L \subseteq A^*$on $A$is regular if it is empty or can be built up from elements of A by using concatenation and the operations + and $*$.

Binary natural numbers can be represented by $1(0 + 1)^*$. Note that any such number must start with a 1, followed by a string of 0's and 1's which may be null [4]. $1(0 + 1)^* = (1, 1(0 + 1), 1((0 + 1)(0 + 1)), 1((0 + 1)(0 + 1)(0 + 1)), \ldots)$

Let $A$be a non-empty finite set. A language $L \subseteq A^{\#}$on $A$is regular if it contains zero $\{0\}$ or can be built up from elements of A by using concatenation.

*Ideals of a Language*

A language $L$ is called a left ideal if $L = A^*L$, similarly $L$ is called a right ideal if $L = LA^*$.

Therefore, a language $L$ is called an ideal if it is both left and right ideal. That is, $L = A^*LA^*$.

A language $L \subseteq A^{\#}$ is called a left ideal if $L = A^{\#}L$, similarly Lis called a right ideal if $L = LA^{\#}$.

Therefore, a language L is called an ideal if it is both left and right ideal. That is, $L = A^{\#}LA^{\#}$.

*Prefix-Closed Language*

Let L be a language over an alphabet $A$. Then $L$is **prefix-closed**if

$$(\forall u \in A^*, v \in A^+)(uv \in L \Rightarrow u \in L),$$

and$L$is closed under taking non-empty prefixes, or more succinctly +-prefix-closed, if

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$$(\forall u \in A^+, v \in A^+)(uv \in L \Rightarrow u \in L).$$

Where, $A$ – an alphabet, $A^+$ – non-empty words over A called the free semigroup.

**Using $A^\#$**

Let $L$be a language over an alphabet $A$in $A^\#$. Then $L$is **prefix-closed**if

$$(\forall u \in A^\#, v \in A^+)(uv \in L \Rightarrow u \in L),$$

and$L$is closed under taking non-empty prefixes, or more succinctly +-prefix-closed, if

$$(\forall u \in A^+, v \in A^+)(uv \in L \Rightarrow u \in L).$$

Where, $A$ – an alphabet, $A^+$ – non-empty words over A called the free semigroup, and $A^\#$ the 0-free semigroup.

**Grammar**

A **Context free grammar (grammar)** $G$ is a quadruple $G = (N, T, P, S)$ where;

$N$is a non-terminal symbols, $T$is a set of terminals, $P$ is a set of production rules and $S$ is a start symbol. The rule is of the form $if a \to b$, then $\alpha a \beta \Longrightarrow \alpha b \beta$ where $\alpha, \beta$ are in $T$. In $n$ steps $\alpha_0 \overset{*}{\Rightarrow} \alpha_n where \overset{*}{\Rightarrow} impliesn - stepsL(G) = \{w \mid w \in T^*: s \Longrightarrow w\}$. Here $T^*$ is a top terminal string.

Consider the grammar $N = \{s\}, T = \{a, b\}, P = \{(1)S \to aSb, (2)S \to ab\}$ . The language generated by this grammar is as follows,

$S \Longrightarrow ab, ab \in L(G)$, $S \Longrightarrow aSb$ using rule (2) we have that, $S \Longrightarrow aabb$where $a^2b^2 \in L$(G).

Using rule (1) we have that $S \Longrightarrow aSb \Longrightarrow aaSbb \Longrightarrow aaabbb$ where $a^3b^3 \in L(G)$

Therefore, the language by this grammar will be of the form; $L(G) = \{a^n b^n \mid n \geq 1\}$

A language that recognizes the context free grammar is called a **Context free language**.

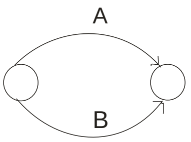
***Automaton***

(i) **Regular Expression And Automaton**

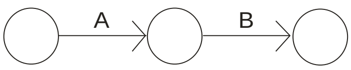
A regular expression is a shorthand form of describing a regular language.

Rules for conversion from regular expression to Automaton

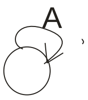
- A regular expression $A \mid B$ "read as a regular expression $AorB$" is




- A regular expression read $AB$ "read as a regular expression $A$followed immediately by the expression$B$" we do this with generally three (3) states.




- $A^*$




This diagram has infinitely many strings of $A's$.

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- The Conversion of the expression $ab^*a$ to a finite state automaton, is as follows;

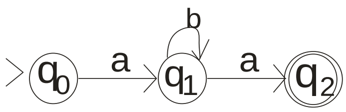

figure 1

where $q_0, q_1, q_2$ are the states of the automaton, $q_0$ is the initial state, and $q_2$ is the final state (or the accept state) of the automaton. The strings that are generated from this state diagram a re; $(aa, aba, abba, abbba, abbbba, abbbbba, \dots)$

(ii) An **Automaton** $\mathcal{A}$ is a triple $(Q, A, \delta)$, where

$Q$ is a finite set of states, $A$ is a finite alphabet, $\delta$ is a transition function $(\delta_i : Q \to Q)_{i \in A}$

Note that the transition function can be interpreted as $\delta: Q \times A \to Q \times A$ .

The state diagram has inputs:

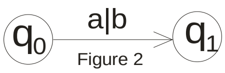

Figure 2

where $a\,|b$ tells us that "$a$" is an input and "$b$" is an output. If we are in state $q_0$ and read symbol $a$, we move to state $q_1$ and output $b$.

That is, $\delta(q_0, a) = (q_1, b)$.

If we are in $q_0$ and read a sequence $\alpha_1\alpha_2 \dots \alpha_n$ we output $\beta_1\beta_2 \dots \beta_n$ where

$\delta(q_{i-1}, \alpha_i) = (q_i, \beta_i)$.

Starting in state $q$ and reading $\alpha$ gives an endomorphism. Extending this to several states gives a homomorphism $\emptyset: Q^+ \to End(A^*)$.Then $\sum(\mathcal{A}) \cong im(\emptyset)$ is the automaton semigroup.

(iii) **A Finite State Automaton *M*** is a 5-tuple $(Q, A, \delta, q_0, F)$, where, $Q$ is a finite set of states, $A$ is a finite set (alphabet) of input symbols, $\delta: Q \times A \to Q$ is the transition function, where $Q \times A$ is the arguments of $\delta$ are state and alphabet symbol, and $Q$ is the result and a state $q_0 \in Q$, is the start state, and $F \subseteq Q$ is the set of accepting, or final states.

A language that accepts the finite state automaton is called a regular language and if otherwise it is non-regular.

***Markov Semigroup***

(i) A **semigroup Markov language** for $S$ over $A$ is a regular language $L$ that does not contain the empty word, is +-prefix-closed, and contains a unique representative for every element of $S$.

(ii) A **robust semigroup Markov language** for $S$ over $A$ is a regular language $L$ that does not contain the empty word, is +-prefix-closed, and contains a uniquerepresentative for every element of $S$ such that $|\,w\,| = \lambda_A(\overline{w})$.

(iii) **The semigroup *S* is *Markov*** (as a semigroup) if there exists a semigroup Markov language for $S$ over an alphabet $A$ representing some generating set for $S$.

(iv) The **semigroup *S* is stronglyMarkov** (as a semigroup) if, for every alphabet $A$ representing a generating set for $S$, there exists a robust semigroup Markov language for $S$ over $A$.

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***Semigroup Automaton***

The automaton $\mathcal{A} = (Q, A, \delta = (\delta_i: Q \to Q)_{i\in A})$, where $Q$ is the finite set of state, $A$ is the finite alphabet and $\delta_i$ is the transition functions.

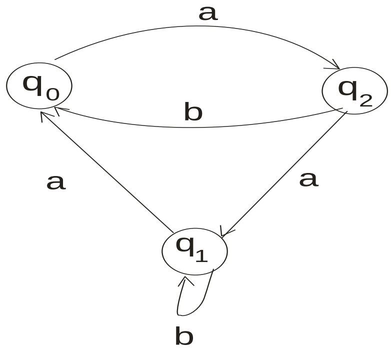


figure 3

The languages generated from figure 3 above are the subsets of $A^+$ *that is*, $(ab, abab, abaa, \dots)$, they are regular, and has unique representatives.

Consider the automaton $\mathcal{A} = (Q, A, \delta = (\delta_i: Q \to Q)_{i\in A}, \rho = (\rho_x: A \to A)_{q\in Q})$, where $\rho_x$ is the production functions

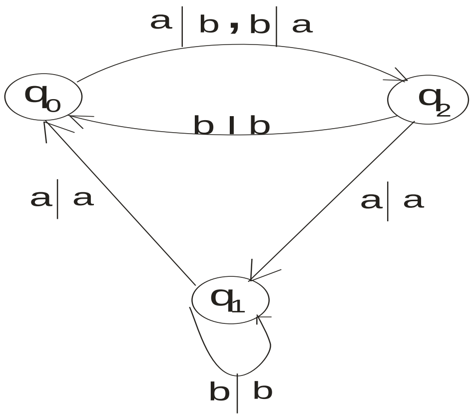


figure 4

A semigroup S is called an automaton semigroup if there exists an automaton $\mathcal{A}$such that $S \backsimeq (\mathcal{A})$.

From figure 4 above, we have as follows;

$\rho(q_0, a) \longmapsto (q_2, b),\ \ \rho(q_2, b) \longmapsto (q_0, b), \rho(q_0, b) \longmapsto (q_2, a),\ \ \rho(q_2, b) \longmapsto (q_0, b),$

$\rho(q_0, a) \longmapsto (q_2, b),\ \ \rho(q_2, a) \longmapsto (q_1, b),\ \rho(q_1, b) \longmapsto (q_1, b),\ \ \rho(q_1, a) \longmapsto (q_0, a).$

Furthermore,

$$Input: \quad a \quad b \quad b \quad b \quad = abbb$$

$$State\ sequence: \quad q_0q_2q_0q_2q_0 \quad = q_0q_2q_0q_2q_0$$

$$Output: \quad b \quad b \quad a \quad b \quad = bbab.$$

Also,

$$Input: \quad a \quad a \quad b \quad a \quad b \quad a \quad b \quad a \quad = aabababa$$

$$State\ sequence: q_0q_2q_1q_1q_0q_2q_1\ \ q_1\ q_0 = q_0q_2q_1q_1q_0q_2q_1q_1q_0$$

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$Output$: $b \quad a \quad b \quad a \quad a \quad a \quad b \quad a \quad = babaaaba.$

For clarity sake, from figure 4 above, we can also see the associative property (moving through $a|b$from $q_0$) as follows;

$$Input: \quad a \quad a \quad a \quad a \quad a \quad a \quad a \quad a \ldots = aaaaaaaa \ldots$$

$$State\ sequence: q_0q_2q_1\ q_0q_2q_1q_0\ \ q_2 \ldots = q_0q_2q_1q_0q_2q_1q_0q_2 \ldots$$

$Output$: $b \quad a \quad a \quad b \quad a \quad a \quad b \quad a \ldots = baabaaba...$

And moving through $b|a$ from $q_0$we have;

$Input$: $b \quad a \quad a \quad b \quad a \quad a \quad b \quad a \ldots = baabaaba ....$

$$State\ sequence: q_0q_2q_1\ q_0q_2q_1q_0\ q_2 \ \ldots = q_0q_2q_1q_0q_2q_1q_0q_2 \ldots$$

$Output$: $a \quad a \quad a \quad a \quad a \quad a \quad a \quad a \ \ldots = aaaaaaaa...$

**SECTION 3**

**Construction Of 0-Free Semigroup.**

The properties of 0-free semigroup $A^{\#}$ are;

- It's a regular language with the adjoined zero as a word that is $L^{\#} = (L^{+} \cup \{0\} \subseteq A^{\#}$
- It is +-prefix-closed and
- It has a unique representation.

**Proposition 3.1**

The 0-free semigroup$A^{\#}$, contains a regular language.

Proof

Let $A$ be a non-empty alphabet and $L$ a language on the alphabet $A$. Suppose that, $A^{\#}$ contains a language $L^{\#}$, then it follows that, for any element $a, b \in A$ we have that $L^{\#} = L^{+} \cup \{0\} = (a, b, ab, bab, . . .) \cup \{0\}$ are elements of $A^{\#}$ noting that the zero in $A^{\#}$ is in isolation from $A^{+}$ (that is $A^{+}$ and $\{0\}$ are disjoint elements of $A^{\#}$ and that $(A^{+})^2 \neq \{0\}$. As a matter of fact $A^{\#} = \{0\} \cup A^{i}$ $(\forall\, i = 1,2,3,. . . n \in \mathbb{N})$this implies that$A^{\#} = \{0\} \cup A^{i} = \{0\} \cup A^{1} \cup A^{2}. . .$These elements built from $A$ must be languages in $A^{\#}$ since $L^{\#} \subseteq A^{\#}$ which are finite or infinite set of strings of $A$. Hence $L^{\#}$ is a language in $A^{\#}$.

Furthermore, suppose that $L^{\#}$ is regular in $A^{\#}$, then it follows that it contains the word $\{0\}$ or can be built up from $A$ by concatenation. Hence $A^{\#}$ contains a regular language $L^{\#}$. Clearly, any language that is recognized by thegivenexpression $L^{\#} = (L^{+} \cup \{0\} \subseteq A^{\#}$ regular.

□

**Proposition 3.2**

A 0-free semigroup$A^{\#}$ has a unique representative.

Proof

Since $A^{\#}$ contains a regular language $L^{\#}$from prop. 3.1. it suffice to show that $A^{\#}$ has a unique representation over $A$ representing the generating set. We write $A^{\#}$ in the form $A^{+} \cup \{0\}$ where 0 represent the adjoined zero and it must be in isolation from $A^{+}$. Also, no product of the elements of $A^{+}$is equals the adjoined zero. Suppose $ab$is a word in $L^{\#}$ with $ab = a0b$ then $a0$ and $b$ represent the zero of $A^{+}$ which contradicts the unique representative

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in $L^{\#}$. But the word $ab$ can only contain a single symbol 0 which must be the unique shortest word over $A^{\#}$. Hence $A^{\#}$ contains a unique representative.

**Proposition 3.3**

$A^{\#}$is +-prefix-closed.

Proof

Since $L^{\#}$ is a regular language in $A^{\#}$. Then $L^{\#} = L^{+} \cup \{0\}$. For all $a \in A^{\#}$and $b \in L^{+}$, then $L^{\#}$ is prefix-closed since $ab \in L^{\#}$ which implies that $a \in L^{\#}$ is a prefix of $ab$. Also $L^{\#}$ is prefix-closed if it is recognized by a finite state automaton in which every state is an accept state [2].Then from 3.2 above, it follows that$L^{\#} = L^{+} - \{a0\}$ is +-prefix-closed.□

## CONSTRUCTION OF MARKOV 0-SIMPLE SEMIGROUP

### 0-Simple Semigroup Automaton

Let $A = \{a, b\}^{\#}$ then we have the automaton $\mathcal{A}$

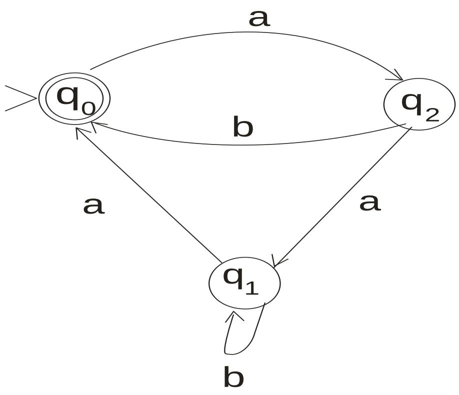


figure 5

The languages that can be recognized by this automaton are the Languages $\{0\}$ that has thesin glesymbol 0 and the free semigroup $A^{+}$, that is, $(ab, abab, aaa, aaba, aabba, \ldots)$.From figur e 5 above, the automaton $\mathcal{A}$ is finite, it contains the word $\{0\}$ with the single symbol 0, it is +-prefix-closed, it has a unique representatives and specifically has a start and end state. The transition function is such that, $Q \times A \to Q$. That is,

$\delta(q_0, a) \to q_2, \quad \delta(q_2, b) \to q_0,$

$\delta(q_0, a) \to q_2, \quad \delta(q_2, a) \to q_1$ and $\delta(q_1, a) \to q_0$.

**Proposition 4.2**

A 0-simple semigroup is Markov.

Proof.

Let $S$ be a 0-simple semigroup and $L$ be a 0-simple Markov language for $S$ with respect to the finite alphabet $A$ representing the generating set$S$, where $L$ is of the form $L \cup \{0\}$ which is a Markov language for $L$.

Consider a deterministic finite state automaton $\mathcal{A}$ with an initial state and every state an accept state, but at this point, we emphasize more on the languages that terminates at $q_0$. This automaton $\mathcal{A}$ must be of the form (as shown) in figure 5 above with every state an accept state. Let $q_0 \in Q$ be an initial state of $\mathcal{A}$, then we obtain a unique representation $\delta(q_0, a) \mapsto q_2$ and $\delta(q_2, b) \mapsto q_0$,

$\delta(q_0, a) \mapsto q_2$, $\delta(q_2, a) \mapsto q_1$and $\delta(q_1, a) \mapsto q_0$. Where $q_0, q_1, q_2 \in Q$ the set of state, $a, b \in A$ the generating set, $\delta$ is the transition function that maps one state to the other through any given path say $a$ or $b$ respectively and ($\{0\}$, $ab$, $aaa$,. . .) are the languages of $\mathcal{A}$ generated by the generating set $A$ for $S$. Since any language accepted or recognized by $\mathcal{A}$ is regular, then $L$ is a regularlanguage for $S$ and is Markov noting that the automaton $\mathcal{A}$ has all of its states as an accept state with unique representatives.

Let $S$ be an ideal, then $S^2 = S$ which by definition must be in isolation from $\{0\}$ as there is no two elements of $S$ that is equals zero, hence $S^2 \neq \{0\}$. Since the initial state of $\mathcal{A}$ is also an accept state then it contains the word $\{0\}$ which is an ideal of $S$. If there exist a zero which is not a prefix of any word $w \in L$ then we have that $w = w_1 0 w_2 = \{0\}$ is a unique word in $L$ that is recognized by $\mathcal{A}$, and is regular, also $w$ and $w_1 0$ represents the ideal $\{0\}$ of $S$. Furthermore, let $w_1, w_2 \in L$ be the left and right ideals generated by $A$ for $S$, then for non-zero elements $a, b \in A$ we have that $b = w_1 a w_2$ is a regular language for $S$ and so $w_1 a$ is a prefix of $b$.

Suppose that $L' = (L - \{0\})$, then $L'$ is a regular language, is +-prefix closed and it maps bijectively onto $S$. Notice that $L' \subseteq A^+$ and is Markov, hence there exists a 0-simpleMarkov language for $S$ over $A$ since $L$ is a regular language, is +-prefix closed, has a uniquerepresentative and contains the word $\{0\}$. Therefore, $S$ is Markov. □

**Proposition 4.3**

A 0-simple semigroup is strongly markov.

Proof,

It follows from 4.2 above. Suppose $S$ is strongly Markov and let $L$ be a language in $S$. Then it follows that there exists a deterministic finite state automaton $\mathcal{A}$ with the initial and final stat eat $q_0$ and every other states being an accept state recognizes the language that contains the w ord with a single symbol 0, hence $L$ is a regular language which is also Markov.
Furthermore, there exist a robustly Markov language $L$ such that, the length of the shortestdist ance over $A$ representing $\overline{w}$ denoted by $|w| = \lambda_A(\overline{w}) = 0$ as a symbol which is regularand is robustly Markov with respect to an alphabet $A$ represe nting a generating set for $S$. Since the generating set for $S$ must be of the form $A \cup \{0\}$, where 0 represent the adjoinedsymbol. Hence $S$ is strongly markov, since there is an exi stence of robust Markov language. □

**The properties obtained as a result of the construction of markov 0-simple semigroup**

- It contains regular languages.
- It contains the word $\{0\}$, since the initial and final state of $\mathcal{A}$ are the same.
- It is +-prefix-closed.
- It contains unique representatives.

**Proposition 4.5**

Let $S$ be a Markov 0-simple semigroup, then

1. It contains the word $\{0\}$. Since the initial and final state of $\mathcal{A}$ are the same.

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2. The generated languages are regular.
3. It is +-prefix-closed.
4. It contains a unique representative.

Proof;

(1) Let$A$ be an alphabet representing a generating set for 0-simple semigroup $S$. Let $\mathcal{A}$ be a deterministic finite state automaton which has an initial state $q_0$ and every state being an accept state, then there exists a new word $\{0\} \notin A$ representing the zero in 0-simple semigroup $S$, that is. Hence S contains the word $\{0\}$.

(2) Suppose $L$ is 0-simple semigroup Markov language for $S$ with respect to the generating set$A$which is of the form $L \cup \{0\}$. Then $L$ is regular since the languages generated are recognized by $\mathcal{A}$.

(3) Let $w \in L$, then there exists $\{0\}$ in $L$which is not a prefix of any word $w$ then we have that $w = w_1 0 w_2 = \{0\}$ is a unique word in $L$ that is recognized by $\mathcal{A}$, is a regular language, also $w$ and $w_1 0$ represents an ideal of $S \neq \{0\}$. Let $w_1, w_2 \in L$ be the left and right ideals generatedby$A$ for $S$, then for non-zero elements $a, b \in A$ we have that $b = w_1 a w_2$ is a regular language for$S$ and so $w_1 a$ is a prefix of $b$.

Suppose that $L' = (L - \{0\})$, then $L'$ is +-prefix closed and it maps bijectively unto $S$.

Noticethat$L' \subseteq A^+$. Hence $L'$ is +-prefix closed.

Finally, (4) follows from (3) above, since $L' \subseteq A^+$ then we have the unique representative for every element of $S$ as follows;

$\delta(q_0, a) \mapsto q_2$and$\delta(q_2, b) \mapsto q_0$,

$\delta(q_0, a) \mapsto q_2$, $\delta(q_2, a) \mapsto q_1$and $\delta(q_1, a) \mapsto q_0$.

where $q_0, q_1, q_2 \in Q$ the set of state, $a, b \in A$ the generating set, $\delta$ is the transition function that maps one state to the other through any given path say $a$ or $b$ respectively and ($ab$, $aaa$,. . .) are the languages of $\mathcal{A}$ generated by the generating set $A$ for $S$. Since any language accepted or recognized by $\mathcal{A}$ is regular, then there exist a regularlanguage $L$ for $S$ and is Markov nothing that the automaton $\mathcal{A}$ has all of its states as an accept state with unique representatives. □

## SECTION 5.

**Theorem 4.1** Let S be a Markov 0-simple semigroup and let T be finite 0-simple semigroup. Then S × Tis a Markov 0-simple semigroup if and only if it is finitely generated.

Proof;

Suppose that $S \times T$ is a Markov 0-simple semigroup, let $L$ be a language in $(S \times T)$ with respect to the finite alphabets $A$ and $B$ representing the generating sets for $S$ which must be of the form $(A \times B) \cup \{0\}$where 0 is the adjoined zero.

Assume that, $S^2 \neq S$ so that $S$ has an indecomposable elements. But the infinitely many element $(s, t)$ $(t \in T)$ is indecomposable in $S \times T$ and hence it belongs to every generating set for $S \times T$ which contradicts the fact that $S \times T$ is finitely generated. Hence $S^2 = S$ and similarly, $T^2 = T$ therefore $S \times T$ is finitely generated. With these properties we have that

$(S \times T)^2 = S \times T$. The converse is also true; if $(S \times T)^2 = S \times T$ then $S^2 = S$ and also $T^2 = T$, since $S$ and $T$ are the homomorphic images of $S \times T$.

Let $A = \{a_i : i \in I\} \cup \{0\}$ and $B = \{b_j : j \in J\} \cup \{0\}$ be generating sets for $S$ and $T$ respectively, choose $s_i \in S$ , $t_j \in T$ $(i \in I, j \in J)(i, j = 1,2, \ldots n)$ and the functions

$$\alpha : I \to I \ and \ \beta : J \to J$$

With $a_i = a_{\alpha(i)} s_i \ \forall \ i \in I \ \ and \ b_j = b_{\beta(j)} t_j \ \forall \ j \in J$ then the set

$$(A \cup \{s_i : i \in I\}) \times (B \cup \{t_j : j \in J\}) \cup \{0\} ------(1)$$

generates $(S \times T)$.

That is, let $s = \sigma_1 \sigma_2 \ldots \sigma_n$ and $= \rho_1 \rho_2 \ldots \rho_n$ , then $S$ and $T$ are written as the products of $n$ elements from $(A \cup \{s_i : i \in I\}) \cup \{0\}$ and $(B \cup \{t_j : j \in J\}) \cup \{0\}$ respectively.

Then we can write $(s, t)$ as product of elements from equation (1) above as

$$(s, t) = (\sigma_1, \rho_1)(\sigma_2, \rho_2) \ldots (\sigma_n, \rho_n) \cup \{0\}.$$

Clearly, the length of $A$ and $B$ are equal, that is, $|A| = |B|$.

Hence $(s, t)$ is finitely generated. Furthermore, from proposition 3.4 we obtain that $(S \times T)$ is markov 0-simple since there is an existence of 0-simple semigroup Markov language $L$ (that is, $(S \times T)$ is a regular language and $L' = L - \{0\}$ is +-prefix-closed and maps onto $(S \times T)$ bijectively). Also, if $B$ is a finite alphabet in bijection with $T$, and $T^2 = T$ then it follows that $T^n = T \ \forall \ n \in \mathbb{N}$ which implies that $T^3 = T^2.T = T.T = T$ and similarly, $S^3 = S^2.S = S.S = S$. Therefore, $(S \times T)$ is markov 0-simple semigroup and is finitely generated. □

## CONCLUSION

The markov 0-simple semigroup $S$ has proven that by the introduction of a word $\{0\}$ with a symbol 0 to the existing markov semigroup [2] we can generate a markov 0-simple semigroup. Hence a 0-simple semigroup is markov. Also, by the introduction of the robustly Markovlanguage to $S$, we obtained that a 0-simple semigroup is strongly markov. Furthermore, two Markov 0-simple semigroups $S$and $T$, produces the direct product $(S \times T)$ that is markov 0-simple. Hence if $S$ is a Markov 0-simple semigroup and $T$ is finite then, weobserve that $(S \times T)$ is a markov 0-simple semigroup if and only if it is finitely generated.


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**REFERENCES**


[1]. Brzozowski, J., Jir´askov´a, G., & Chenglong, Z.D.R. (2001).Quotient Complexity of Closed Languages.Mathematical Institute, *Slovak Academy of Sciences,6*, 040 01.

[2]. Cain, A. J., & Maltcev, V. (2014). Markov semigroups, monoids, and groups. *Internat ional Journal of Algebra and computation, .24*(05); pages 609-653.

[3]. Campbell, C.M., Robertson, E.F., Ruskuc, N., & Thomas, R.M. (2000). Direct produc ts of automatic semigroups, *J. austral. Math.soc.ser.A69* (1) p 19-24.

[4]. Chen, W.W.L. (1991, 2008). *Discrete Mathematics*. Lecture note.

[5]. Harju, T. (1996).*lecture note on Semigroup*. Department of Mathematics University of Turku FIN-20014 Turku, Finland

[6]. Howie, J.M. (1995). *Fundamentals of Semigroup Theory, Vol. 12 of London Mathematical Society Monographs (New Series)*, Clarendon Press, Oxford University Press, New York.

[7]. Jean-´Eric, P. (2014). *Mathematical Foundations of Automata Theory*. Uncompleted textbook.

[8]. Klimann, I. (2012). Automaton semigroup, LIAFA-UMR7089 CNRS &Universit`e Paris Diderot MealyM-ANR JCJC 12 JS0 01.

[9]. Krithivasan, K. (2012). *Theory of computation; Grammar and Languages*.YouTube.

[10]. Robertson, E. F., Ruskuc, N., & Wiegold, J., (1998). Generators and relations of direct products of semigroups, *Trans. Amer. Math.Soc. 350* (7) p 2665-2685.

[11]. Ruskuc, N., Araujo, I.M., Robertson, E.F., & Wiegold, J. (2003). Generators and Defining Relations for Direct Products of Semigroups.


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