Graph Expansion of 0-Simple Semigroup

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ABSTRACT

We use graph expansion to construct 0-simple Semigroup. Properties arising from this construction are highlighted. The graph expansion of 0-simple Semigroup presentation gives a pictorial view of the constructed 0-simple Semigroup and enhances the exposition of it's properties.

Keywords: Graph expansion, 0-simple semigroup,
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INTRODUCTION

This work used graph expansion to construct a class of semigroup called 0-simple semigroup. We recall that a Semigroup, S is called 0-simple if

(i) S and $\{0\}$ are it's only ideals and

(ii) $S^2 \neq \{0\}$,

that is, 0-simple Semigroup S is a semi group that has no proper ideal except itself and zero. The use of graph has become widespread in the algebraic theory of Semi groups. The graph is mainly used as a visual, to make presentations clearer and the problems more imaginable. Central to this approach is the Cayley graph of Semi groups [6]. Cayley graph is a graph that encodes the structure of a semigroup. It uses a specified, usually finite set of generators. It is a method of presentation of a group where the vertices of the graph are the elements of the semigroup called the generators of the semigroup. The technique of using Cayley graph of a group presentation for construction is called Graph Expansion. Margolis and Meakin [10] used Cayleygraph of a group presentation to construct E-unitary inverse monoids which belong to the quasi variety of weakly left ample monoids. The basic philosophy of this work and also that of [3], [4], [5], [7] and [10] is the novel idea of construction using Cayley graph known as graph expansion. Gomes and Gould [3] considered graph expansionof unipotent monoids, the monoids arising from this were E-unitary and belonged to the quasi variety of weakly left ample monoids. Heale [7] constructed an expansion from the Cayley graph of a Semigroup, S and proved that if S is E-dense then so is it's expansion and they have the same universal group. Graham [5] employed technique which involved the transformation of the algebraic problems into equivalent graph-theoretic problem by means of natural correspondence between subsets of the semigroup and subgraphs of a certain directed bipartite graph. Gould [4] used Cayley graph of a right cancellative monoid to construct left adequate monoids. Definitions, concepts and terms vital in the paper are given in Section 2. In section3, we considered a 0-simple Semigroup presentation (X, f, S) where X is non-empty and finite, the generator of the 0-simple Semigroup and X: f $\rightarrow S$ generates Xf as a 0-simple Semigroup. This section also gives the construction of the 0-simple Semigroup where the elements of the 0-simple Semigroup were represented in the graph. Section 4 highlights the properties from the constructed 0-simple semigroup and some practical applications are given.

PRELIMINARIES

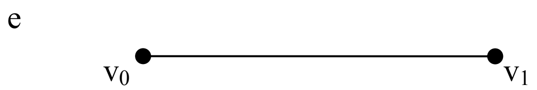
In this Section, we are going to discuss concepts that will be used in the whole work. However, for terminologies and definitions, we will rely on [1], [2]and [9]

Graph

i. A graph G here would be regarded as a pair consisting of a set V = V(G) the set of vertices, E = E(G), the set of edges and together with the maps

$$i:\ E \longrightarrow V \text{ and } t:\ E \longrightarrow V$$

Here, irepresents the initial map and t, the terminal map. As an example, if E = {e} and V={ v_0,v_1}. Pictorially,



Then the map form

$$i:\ E \longrightarrow V \text{ and } t:\ E \longrightarrow V$$

will be defined as

$$i\,(e) = v_0 \text{ and } t\,(e) = v_1$$

Remark : In this work, an edge will be represented by a pair of vertices (v_0,v_1) or $v_0v_1$or e.
Remark : If two vertices are not joined together, then the edge is empty or zero, that is, there is no edge.

Definition

(i) Walk: A walk W on a graph G is a sequence of vertices and edges W = $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, \ldots v_{n-1}, e_{n-1}, v_n$where v_i's are the vertices and e_i's are the edges such that each $e_i$is an edge betweenv_i and v_{i+1}.The vertex $v_1$is called the initial vertex and the terminal vertex v_n. If $v_1 = v_n$, W is called a closed walk. A closed walkis a walk that starts and ends at the same vertex.

ii. Null Walk: A null walk is a walk of length zero '0'.

iii. Loop: A loop is an edge from a vertex back to the same vertex

iv. Trail: A trail is a walk whose edges are distinct, that is, no edge is repeated.

v. Path: A path in a graph, G, is a sequence of k ≥ 0 vertices $v_0, \ldots, v_k$ such that $v_i \longrightarrow v_{i+1}$ is an edge for all i where $0 \leq i < k$. The path is said to start at v_0 and ends at v_k, the length of the path is defined to be k. In order words, apath is a walk whose vertices and edges are distinct, except the initial and terminal vertices.

vi. An empty path I_v is a path with one vertex and no edge.

vii. Cycle:- A cycle is a path that begins and ends with the same vertex. A cycle with n vertices is called an n-cycle. In order words, a cycle is a closed path whose initial and terminal vertices are identical. A graph with no cycle in it is called acyclic.

viii. Tree :- A tree is a connected graph that has no cycles, hence, tree is a connected acyclic graph.

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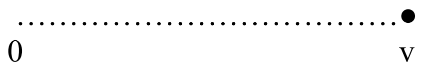
ix.0-vertex:- A 0-vertex is a vertex with no edge incidence to it.



x. Isolated vertex:- An isolated vertex is a vertex that has degree zero.

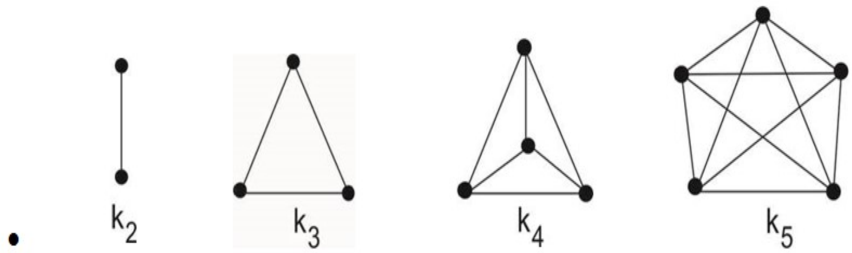
xi. 0-rooted graph :- A graph G is 0-rooted when it has a null walk or empty path. Hence, a graph is 0-rooted where 0 ∈ V for all w ∈ V, there is a path with 0 edge from 0 to w. However, this path has null walk.

Simple Graph

A simple graph is an undirected graph that has no loop and not more than one edge between any two different vertices. The edges of the graph form a set and each edge is a distinct pair of vertices.

Complete Graph

Each pair of vertices is joined by an edge. The graph contains all possible edges. A complete graph k_n of order n is a simple graph with n vertices in which every vertex is adjacent to another vertex. Complete graphs with n vertices have $\binom{n}{2}$ edges.



Complete graph k_n at n= 1, 2, 3, 4 and 5

Primitive Idempotent

Let S be a semigroup with zero, then the defining properties of a zero element immediately imply that '0' is the unique minimum idempotent. The idempotent that are minimal within the set of non-zero idempotents are called primitive.

Completely 0-Simple Semigroup

Let S be a semigroup with zero, then S is called completely 0-simple if it is 0-simple and has a primitive idempotent.

CONSTRUCTION

This section involves the construction of the 0-simple semigroup using graph expansion. Here, the elements of the 0-simple semigroup are represented as vertices of our graph. The graph considered in this work is a simple graph with no loop except stated otherwise. A graph G consists of two sets V(G), the set of vertices and E = E(G) , the set of edges and together with

$$i : E \rightarrow V \quad \text{and} \quad t : E \rightarrow V$$

wherei and t are the initial and terminal maps respectively. Let G be a graph, Δ_G will denote a subgraph of G if $\Delta_G \subseteq$ G. In the graph G, e will denote an edge, if v_1, v_2 are the vertices of graph G, then i(e) = v_1 and t(e) = v_2

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Generally, if e_1 , e_2, . . . are the edges of G and v_1, v_2, . . . are the vertices then the initials and terminals of each of the edges are

i(e_1) = v_1 , t(e_1) = i(e_2) , t(e_2) = i(e_3),. . . t(e_2) $= w$

Thus, a path from a vertex v to a vertex w is a finite sequence of edges e_1 , e_2, . . . ,e_n with

i(e_1) = v_1 , t(e_1) = i(e_2) , t(e_2) =i(e_3),. . . ,t(e_n) $= w$

A subgraph Δ_G of G consists of subsets V_Δ of V_G and E_Δ of E_G such that for any e ∈ E_Δ

$$i(e_1) \ , \ t(e_1) \in V_\Delta$$

This path determines a subgraph. A graph morphism θ from a graph G to a graph G' consists of two functions each denoted by θ from V_G to $V_{G'}$ and from E_G to $E_{G'}$ such that for any e ∈ E_G

i(e) θ = i(eθ) and t(e)θ = t(e θ)

so for $i_1(e_1)$, $i_2(e_2)$ ∈ E_G

$$[i_1(e_1) \,.\, i_2(e_2)\ ]\,\theta = [i_1(e_1\theta) \,.\, i_2(e_2\theta)\,]$$
$$= [i_1(e_1)]\theta \,.\, [\,i_2(e_2)]\,\theta$$

The Zero Element of the Graph

We define the action of a 0-simple semigroup S on a graph G. A 0-simple semigroup S acts on a graph G if V_G and E_G are left S-sets and i and t are left S-maps that is

i(se) = si(e) and t(se) = st(e) for all s ∈ E_G.

Hence, the action of any s ∈ S is graph morphism so that $\Delta_G$is a subgraph of G and also sΔ_G is a subgraph. Since our graph is a simple graph, it is enough to just give an ordered list of vertices that the path is to transverse. Since 0-simple semigroup is a finite semigroup, the graph considered in this work is the complete graph k_n with n vertices for n ≥ 3. We define a complete graph $k_n$with n vertices for n ≥ 3 and labelling edges thus:

For the purpose of this work, we define a 0-vertex as a vertex from 0 to V with empty edge having only one vertex.

-----------------●
0 v

Hence e_1= 0 ⇒ there is no edge. For v_i, v_{i+1} =e_i, e_i =0 for only i =1. This ascertains the existence of 0 in the graph, so, corresponding to each x ∈ X, there is an edge e_i , with $e_i = 0$ for only i =1

0-Simple Semigroup Presentation

Here, we consider a 0-simple semigroup presentation (X, f, S) whereX is finite , non-empty set, S, a 0-simple semigroup and f: X→S, f is a function such that Xf generates S as a 0-simple semigroup. Here, V_G= S and $i(e_1)$= 0, if e_i= 0 for only i =1, $t(e_i)$ = s if s ∈ S. Hence the edge set

E_G= { (s, x, s(xf) : s ∈ S and x ∈ X}

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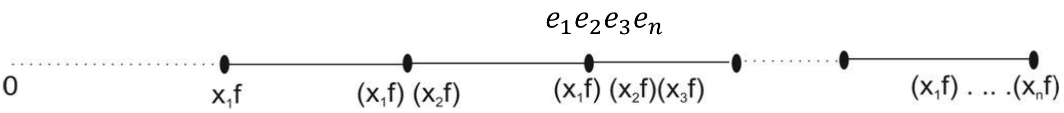
where

$i(s, x, s(xf)) = 0$ if $s = 0$

$i(s, x, s(xf)) = s$ if $s \neq 0$ and

$t(s, x, s(xf)) = s(xf)$ if $s \in S$



Let 0-simple semigroup S act on G where $s \in S$ and $(s, x. s(xf)) \in E_G$ we have note that for $(t, x, t(xf)) \in E_G$, $s \in S$

$$s.(t, x, t(xf)) = (st, x, st(xf)).$$

Considering $\Delta_{G,}$the subgraph of G, the graph expansion is defined thus:

M = M (X, f, S) of (X, f ,S) a 0-simple semigroup presentation is expressed thus

$M = \{ (\Delta_G, s) : \Delta_G$ is a finite 0-rooted subgraph of G and $0, s \in V_{\Delta_G}\}$.

We now define a multiplication on Mby

$$(\Delta_G,\ s)(\Sigma_G,\ t) = \begin{cases} (\Delta_G \cup s\Sigma_G, st) & if\ s,t \neq 0 \text{ and } s.t \neq 0 \\ (\Delta_G, 0) & if\ \ s\ or\ t = 0, s.t = 0 \end{cases} \qquad \text{(i)}$$

$(\Delta_G, a)\,0 = 0\,(\Delta_G, a) = (\Delta_G, 0) = 0$, $a \in \Delta_G$

We identify all the element of the form $(\Delta_G, 0)$ with 0 under the multiplication given.

Graph Expansion of 0-Simple Semigroup

Theorem 3.2

Let (X, f, S) be a 0-simple semigroup presentation where X is a set, S is a 0-simple semigroup

generated by Xf where $f: X \longrightarrow S$. The graph expansion (X, f, S) is a 0-simple semigroup with

multiplication as defined in (i).

Proof

a. Closure

Let $(\Delta_G, p), (\Sigma_G, q) \in M$ (X, f, S)

$$(\Delta_G, p)(\Sigma_G, q) = \begin{cases} (\Delta_G \cup p\Sigma_G, pq) & if\ p,q \neq 0\ and\ p.q \neq 0 \\ (\Delta_G, 0) & if\ \ p\ or\ q = 0, p.q = 0 \end{cases}$$

In $(\Delta_G \cup p\Sigma_G, pq)$, Δ_G is a subgraph of G and so is $p\Sigma_G$ hence $\Delta_G \cup p\Sigma_G$ is also a subgraph and pq is a vertex in the graph .By the definition of 0-vertex, $(\Delta_G, 0)$ is the zero of the graph and '0' has been defined in the graph G hence Δ_G is a subgraph and $(\Delta_G, 0)$ is the zero element in Δ_G so, the multiplication is well defined.

b. *Associativity*

A semigroup is an associative groupoid, so, confirming that with the multiplication defined in (i). M (X, f, S) is associative

Let (Δ_G, p), (Σ_G, q), (ζ_G, r) ∈M (X, f, S) and p, q, r ∈ S

To show that

$(\Delta_G, p)\ \{(\Sigma_G, q)\ (\zeta, \mathrm{r})\} = \{(\Delta_G, p)\ (\Sigma_G, q)\}\ (\zeta_G, \mathrm{r})$

LHS :$(\Delta_G, p)\ \{(\Sigma_G, q)\ (\zeta_G, \mathrm{r})\} = (\Delta_G, p)\ \{(\Sigma_G \cup q\zeta_G, \mathrm{qr})\}$

$= \{\Delta_G \cup p(\Sigma_G \cup q\zeta_G), \mathrm{p(qr)}\}$

$= (\Delta_G \cup p\Sigma_G \cup pq\zeta_G, \mathrm{pqr})$

RHS $\{(\Delta_G, p)\ (\Sigma_G, q)\}\ (\zeta_G, \mathrm{r}) = \{(\Delta_G \cup \mathrm{p}\Sigma_G, \mathrm{pq})\}\ (\zeta_G, \mathrm{r})$

$= \{\Delta_G \cup \mathrm{p}\Sigma_G \cup \mathrm{pq}\zeta_G, \mathrm{pq(r)}\}$

$= (\Delta_G \cup \mathrm{p}\Sigma_G \cup \mathrm{pq}\zeta_G, \mathrm{pqr})$

So, LHS =RHS

Hence, the graph expansion M (X, f, S) with the multiplication defined is associative hence a semigroup.

Considering $(\Delta_G, 0)$ the zero element in M (X, f, S)

Let $(\Delta_G, 0)$,$(\Sigma_G, q)(\zeta_G, r)$ ∈M (X, f, S) and 0,q,r ∈ S

To show that

$(\Delta_G, 0)\ \{(\Sigma_G, q)\ (\zeta, \mathrm{r})\} = \{(\Delta_G, 0)\ (\Sigma_G, q)\}\ (\zeta_G, \mathrm{r})$

LHS:$(\Delta_G, 0))\ \{(\Sigma_G, q)\ (\zeta_G, \mathrm{r})\} = (\Delta_G, 0)\ \{(\Sigma_G \cup q\zeta_G, \mathrm{qr})\}$

$= \{\Delta_G \cup 0(\Sigma_G \cup q\zeta_G), 0(\mathrm{qr})\}$

$= (\Delta_G \cup 0\Sigma_G \cup 0q\zeta_G, 0)$

$=(\Delta_G \cup 0 \cup 0, 0)$

$=(\Delta_G, 0)$

RHS $\{(\Delta_G, 0)\ (\Sigma_G, q)\}\ (\zeta_G, \mathrm{r}) = \{(\Delta_G \cup 0\Sigma_G, 0\mathrm{q})\}\ (\zeta_G, \mathrm{r})$

$= \{\Delta_G \cup 0\Sigma_G \cup 0\mathrm{q}\zeta_G, 0\mathrm{q(r)}\}$

$= (\Delta_G \cup 0\Sigma_G \cup 0\mathrm{q}\zeta_G, 0)$

$=(\Delta_G \cup 0 \cup 0, 0)$

$= (\Delta_G, 0)$

c. *The Ideals of M (X, f, S)*

Recall that A set I ⊆ M is an ideal if IM⊆ I. Let M be as defined above, we suppose I is an ideal of M, then IM⊆ I.

$$\mathrm{I\,M} = \{(\Delta_G, t)(\Sigma_G, m):\ (\Delta_G, t) \in I, (\Sigma_G, m) \in \mathrm{M}\}$$

$=\{(\Delta_G \cup t\Sigma_G, tm):(\Delta_G, t) \in I, (\Sigma_G, m) \in \mathrm{M}\}$

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Clearly, $(\Delta_G \cup t\Sigma_G\,, tm) \geq (\Delta_G, t)$ since $\Delta_G \cup t\Sigma_G$ is a subgraph of M, hence $(\Delta_G \cup t\Sigma_G\,, tm) =$ M so, I M = M . Hence, there cannot be any other ideal in M other than M. Similar deduction follow for the left ideal MI. So,M is a two sided ideal.

0 M $= \{ (\Delta_G, 0)(\Sigma_G\,, m) \colon (\Delta_G, 0) \in\ I, (\Sigma_G\,, m)\ \in$M $\}$

$=\{(\Delta_G \cup 0\Sigma_G,\ 0m)\ \colon (\Delta_G, 0) \in\ I, (\Sigma_G\,, m)\ \in$M $\}$

$=\{(\Delta_G \cup 0,\ 0)\ \colon (\Delta_G, 0) \in\ I, (\Sigma_G\,, m)\ \in$M $\}$

$=\{(\Delta_G\,, 0)\ \colon (\Delta_G, 0) \in\ I, (\Sigma_G\,, m)\ \in$M $\}$

But $(\Delta_G, 0)$ is the zero of M (X, f, S) which also belong to M (X, f, S) hence, 0 M $=\{0\}$. So, the only two ideals of M (X, f, S) are itself and $(\Delta_G\,, 0) = \{0\}$.

d. *To Verify* $M^2 \neq 0$

Let (Δ_G, m),$(\Sigma_G\,, t)$ $\in$M,

$M^2 = \{(\Delta_G, m)\ (\Sigma_G\,, t)\ \ \colon (\Delta_G, m), (\Sigma_G\,, t)\ \in$M $\}$

$=\{(\Delta_G \cup m\Sigma_G\,, mt)\ \ \colon (\Delta_G, m), (\Sigma_G\,, t)\ \ \in$M $\}$

Clearly,

$$(\Delta_G \cup m\Sigma_G\,, mt) \neq (\Delta_G\,, 0)$$

Hence $\Rightarrow \{\ (\Delta_G \cup m\Sigma_G\,, mt)\ \ \colon (\Delta_G, m), (\Sigma_G\,, t)\ \ \in$M $\} \neq\ \{0\}$

Hence $\mathrm{M}(X, f, S)$ with the multiplication defined is a 0-simple semigroup.

Hence the graph expansion with the multiplication defined is associative and hence a 0-simple semigroup.

PROPERTIES FROM THE CONSTRUCTION

Theorem 4.1

The graph expansion $\mathrm{M}(X, f, S)$ is completely 0-simple.

Proof

Since $\mathrm{M}(X, f, S)$ is 0-simple and is a finite 0-rooted subgraph of G, then

Proposition 4.2 [9]

In every finite semigroup, there is at least one idempotent.

ii. The idempotent in M (X, f, S)

Let $(\Delta_G, 0) \in$ M, then

$(\Delta_G, 0)^2 = (\Delta_G, 0)(\Delta_G, 0) = (\Delta_G \cup\ 0\Delta_G, 00) = (\Delta_G \cup\ 0, 0) = (\Delta_G, 0)$

$(\Delta_G, 0)$ is the idempotent element in M by the ordering stated in the graph formation

e_i =0 for only i =1. The idempotent is nilpotent. Since M is a semigroup with zero, then the defining properties of the zero element $(\Delta_G, 0)$ immediately imply that $(\Delta_G, 0)$ is the unique minimum idempotent, also Mis finite 0-rooted then $(\Delta_G, 0)$ is the primitive idempotent hence M is completely 0-simple.

Theorem 4.3

Let M (X, f, S) be a 0-simple semigroup presentation, M is 0-simple semigroup, acyclic containing at most one vertex with zero in-degree,

Proof

M is vertex-transitive with the multiplication defined, edge –transitive by the orientation of it's root. Each vertex in M forms a separate strongly connected component. M cannot have more than $\binom{n}{2}$ edges since it's a complete graph, this bound attained by assigning a different value to each vertex and orienting all edges so that the orientation starts from the root 0. Hence, each $|v_i|$ is bounded, then the vertex must be transversed eventually to a vertex already visited, hence there is a cycle, which is not possible since M is acyclic. Mis finite since it's generators are finite hence, there sequence of vertices restart at the root of M. Hence M has at most one vertex with zero in- degree (0-vertex).

Application

Water Park

Water always flow from high altitudes to low altitudes so that when travelling along water slides, one can never arrive at the same spot twice, similitude of acyclic structure in graph.

Growth System of Human Beings

The root of the0-simple semigroup (vertex 0) is synonymous with growth system of human being at birth which is a little static. Being born does not guarantee growth but as soon as growth is activated (0-vertex), it progresses until growth system collapses (peak of the tree of the 0-simple semigroup) from which the growth system deteriorates thereby restarting back from the root (root of the 0-simple semigroup). At this stage, if growth cannot be sustained (which definitely cannot happen), there is death, but growth could be reactivated medically for the process to start again, else death. Implying, that each vertex, edge is visited once (Euler path).

CONCLUSION

The technique of using Cayley graph method for construction has been used to construct 0-simple semigroup which is completely 0-simple. Applications of graphs of 0-simple semigroup in graph were examined and examples given. This work adds to the growing information on graphs of 0-simple semigroup.

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