ISSN: 2186-8476, ISSN: 2186-8468 Print
Vol. 2 No. 2, June 2013

SOME SPECIALIZATIONS AND EVALUATIONS OF THE TUTTE POLYNOMIAL OF A FAMILY OF GRAPHS

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ABSTRACT

In this paper, we give some specializations and evaluations of the Tutte polynomial of a family of positive-signed connected planar graphs.

First of all, we give the general form of the Tutte polynomial $T_G(x, y)$ of the family of graphs G using directly the deletion-contraction definition of the Tutte polynomial.

Then, we give general formulas of Jones polynomials $V_L(t)$ of very interesting families of alternating knots and links L that correspond to these planar graphs; we actually specialize the Tutte polynomial to the Jones polynomial with the change of variables, $x \to -t$ and $y \to -t^{-1}$, and with some factor of t. In case of two-component links, we get two different formulas of the Jones polynomial, one when both the links are oriented either in clockwise or counterclockwise direction and another one when one component is oriented clockwise and the second counterclockwise.

Moreover, we give general forms of the flow, reliability, and chromatic polynomials of these graphs. The reason to study flow polynomial is that it gives the number of proper flows in the connected graph, G. In our case, we give the number of nowhere zero flows in G over a finite abelian group K using the Tutte polynomial. The reliability polynomial gives the probability of a path of active edges between each pair of vertices. The chromatic polynomial, which is a popular graph invariant, actually could count the number of ways of proper coloring of the graph. For better understanding of the situation, we also give graphs for all these polynomials for different values of the parameters.

Finally, we give some useful combinatorial information about these connected graphs, particularly about the subgraphs and the orientations of these graphs. Regarding subgraphs, we give the number of subgraphs, number of connected spanning subgraphs, number of forests, and number of trees of these graphs. Regarding orientations, we give the number of acyclic orientations, number of acyclic orientations with exactly one predefined source, number of totally cyclic orientations, and the number of score vectors of orientations of the graph.

Keywords: Tutte polynomial, Jones polynomial, Flow polynomial, Reliability polynomial, Chromatic polynomial

INTRODUCTION

The Tutte polynomial was introduced by W. T. Tutte in 1954 in [19] as a generalization of chromatic polynomials studied by Birkhoff [1] and Whitney [23]. This graph invariant became popular because of its universal property that any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it, and because of its applications in computer science, engineering, optimization, physics, biology, and knot theory.

In 1985, V. F. R. Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras [11]. However, in

1987 L. H. Kauffman introduced in [13] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple; we actually follow this construction.

Our primary motivation to study the Tutte polynomial came from the remarkable connection between the Tutte and the Jones polynomials that up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t, -t^{-1})$ [17, 15, 8, 9].

This paper is organized as follows: In Section 2 we will give some basic notions about graphs and knots along with definitions of the Tutte and the Jones polynomials. Moreover, in this section we will give the relation between graphs and knots, and the relation between the Tutte and the Jones polynomials. In Section 3 the general formula of the Tutte polynomial of a family of graphs will be given. Moreover, in this section we will specialize the Tutte polynomial to the Jones, flow, reliability, and chromatic polynomials. Finally, the interpretations of some evaluations of the Tutte polynomial will be given at the end of the same section.

PRELIMINARY NOTIONS

Basic Concepts of Graphs

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of unordered pairs of$V$. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph, then $V = V(G)$ is the vertex set of$G$, and $E = E(G)$ is the edge set. An edge (x, y) is said to join the vertices x and$y$, and is denoted byxy; the vertices x and y are the end vertices of this edge.

If $xy \in E(G)$ then x and y are *adjacent* or *neighboring* vertices of , and the vertices x and y are *incident* with the edge xy. Two edges are adjacent if they have exactly one common end vertex.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write$G' \subset G$. If G' contains all edges of G that join two vertices in V' then G' is said to be the subgraph induced or spanned by V', and is denoted by $G[V']$. Thus, a subgraph G' of G is an induced subgraph if $G' = G[V(G')]$. If $V = V'$ then G' is said to be a *spanning subgraph* of G.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus, $G = (V, E)$ is isomorphic to $G' = (V', E')$, denoted $G \simeq G'$, if there is a bijection $\varphi : V \to V'$ such that $xy \in E$ if and only if $\varphi(xy) \in E'$.

The dual notion of a cycle is that of cut or cocycle. If $\{V_1, V_2\}$ is a partition of the vertex set, and the set C, consisting of those edges with one end in V_1 and one end in V_2, is not empty, then C is called a *cut*. A *cycle* with one edge is called a *loop* and a cocycle with one edge is called a *bridge*. We refer to an edge that is neither a loop nor a bridge as *ordinary*.

A graph is *connected* if there is a path from one vertex to any other vertex of the graph. A connected subgraph of a graph G is called the *component* of G. We denote by $k(G)$ the number of connected components of a graph G, and by $c(G)$ the number of nontrivial connected components, not counting isolated vertices.

ISSN: 2186-8476, ISSN: 2186-8468 Print
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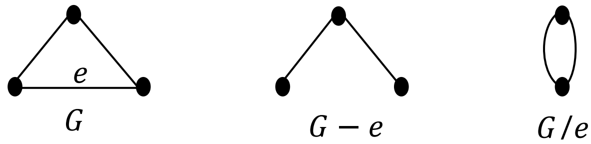
A *tree* is a connected graph without cycles. A *forest* is a graph whose connected components are all trees. (Spanning trees in connected graphs play a fundamental role in the theory of the Tutte polynomial.) Observe that a loop in a connected graph can be characterized as an edge that is in no spanning tree, while a bridge is an edge that is in every spanning tree.

A graph is *planar* if it can be drawn in the plane without edges crossings. A drawing of a graph in the plane separates the plane into regions called *faces*. Every plane graph G has a dual graph, G^* , formed by assigning a vertex of G^* to each face of G and joining two vertices of G by k edges if and only if the corresponding faces of G share k edges in their boundaries. G^* is always connected. If G is connected, then $(G^*)^* = G$. If G is planar, it may have many dual graphs.

A graph invariant is a function f on the collection of all graphs such that $f(G_1) = f(G_2)$ whenever $G_1 \simeq G_2$. A graph polynomial is a graph invariant where the image lies in some polynomial ring.

The Tutte Polynomial

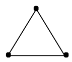
The following two operations are essential two understand the Tutte polynomial definition for a graph G. These are the edge deletion, which is denoted by $G - e$, and the edge contraction, which is denoted by G/e.

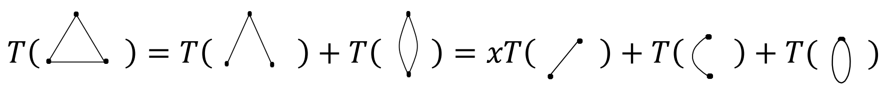


Definition 2.1 [8, 9, 19, 20, and 21] The *Tutte polynomial* of a graph G is a two-variable polynomial $T_G(x, y)$ defined as follows:

$$T_G(x,y) = \begin{cases} xT(G/e) & if\ e\ is\ a\ bridge \\ yT(G-e) & if\ e\ is\ a\ loop \\ T(G-e) + T(G/e) & if\ e\ is\ neither\ a\ bridge\ nor\ a\ loop \end{cases}$$

If E is empty, then $T_G(x, y)$ is 1.

Example. Here is the Tutte polynomial of the graph G:  .



$$= x^2T(.) + xT(.) + y = x^2 + x + y$$

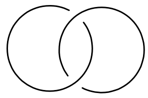
Remark 2.2. The definition of the Tutte polynomial outlines a simple recursive procedure to compute it, but the order of the rules applied is not fixed.

Basic Concepts of Knots

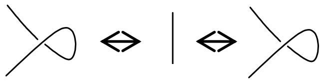
A *knot* is a circle embedded in three-dimensional space, and a *link* is an embedding of a union of such circles. Since knots are special cases of links, we shall often use the term link for both knots and links. Links are usually studied via projecting them on a plan; a projection with extra information of overcrossing and undercrossing is called the *link diagram*.

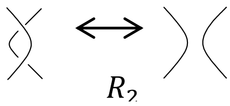

overcrossing
undercrossing

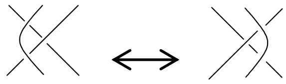

Trefoil knot


Hopf link

Two links are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result about the isotopic link diagrams is: Two unoriented links L_1 and L_2 are equivalent if and only if a diagram of L_1 can be transformed into a diagram of L_2 by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves of the following three types:


R_1


R_2


R_3

The set of all links that are equivalent to a link L is called a *class* of L. By a link L we shall always mean a class of the link L.

The Jones Polynomial

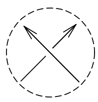
The main question of knot theory is *which two links are equivalent and which are not*? To address this question one needs a knot invariant, a function that gives one value on all links that belong to a single class and gives different values (but not always) on links that belong to different classes. In 1985, V. F. R. Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras [11]. However, in 1987 L. H. Kauffman introduced in [13, 14] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple.

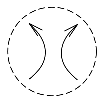
Definition 2.3. [11, 12, 13] The *Jones polynomial* $V_L(t)$ of an oriented link L is a Laurent polynomial in the variable $\sqrt{t}$ satisfying the skein relation

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)V_{L_0}(t)$$

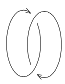
and that the value of the unknot is 1. Here L_+, L_- and L_0 are three oriented links having diagrams that are isotopic everywhere except at one crossing where they differ as in the figure below:


L_+


L_-


L_0

Example. The Jones polynomials of the Hopf link and the trefoil knot are respectively

$V($  $) = -t^{-1/2} - t^{-5/2}$ and $V($  $) = -t^{-4} + t^{-3} + t.$

ISSN: 2186-8476, ISSN: 2186-8468 Print
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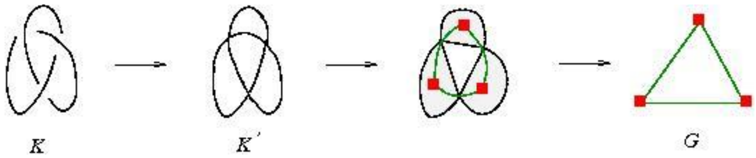
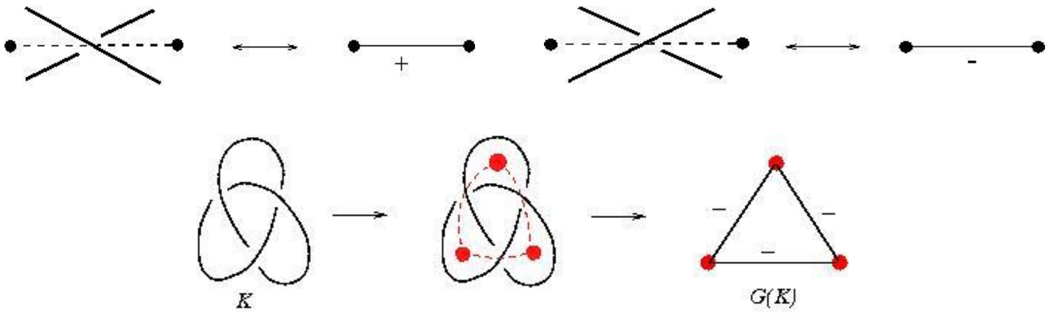
A Connection between Knots and Graphs

Corresponding to every connected link diagram we can find a connected signed planar graph and vice versa. The process is as follows: Suppose K is a knot and K' its projection. The projection K' divides the plane into several regions. Starting with the outermost region, we can color the regions either white or black. By our convention, we color the outermost region white. Now, we color the regions so that on either side of an edge the colors never agree.



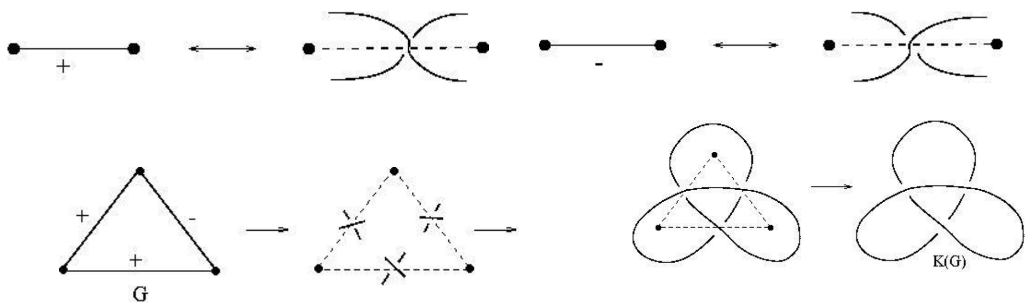
Next, choose a vertex in each black region. If two black regions R and R' have common crossing points $c_1, c_2, \cdots, c_n$ then we connect the selected vertices of R and R' by simple edges that pass through $c_1, c_2, \cdots, c_n$ and lie in these two black regions. In this way, we obtain from K' a plane graph G [16].

However, in order for the plane graph to embody some of the characteristics of the knot, we need to use the regular diagram rather than the projection. So, we need to consider the under- and over-crossings .To this end, we assign to each edge of G either the sign $+$ or $-$ as you can see in the following figure.



A signed plane graph that has been formed by means of the above process is said to be the graph of the knot K [16].

Conversely, corresponding to a connected signed planar graph, we can find a connected planar link diagram. The construction is clear from the following figure.



The fundamental combinatorial result connecting knots and graphs is:

Theorem 2.4. [15] *The collection of connected planar link diagrams is in one-to-one correspondence with the collection of connected signed planar graphs.*

Connection between the Tutte and the Jones polynomials

The primary motivation to study the Tutte polynomial came from the following remarkable connection between the Tutte and the Jones polynomials.

Theorem 2.5. (Thistlethwaite's) [17, 15, 9] *Up to a sign and multiplication by a power of t the Jones polynomial* $V_L(t)$ *of an alternating link* L *is equal to the Tutte polynomial* $T_G(-t, -t^{-1})$.

For positive-signed connected graphs, we have the precise connection:

Theorem 2.6. [2] *Let* G *be the positive-signed connected planar graph of an alternating oriented link diagram* L *Then the Jones polynomial of the link* L *is*

$$V_L(t) = (-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}} T_G(-t, -t^{-1}),$$

where $a(L)$ *is the number of vertices in* G, $b(L)$ *the number of vertices in the dual of* G, *and* $wr(L)$ *the writhe of* L.

Remark 2.7. In this paper, we shall compute Jones polynomials of links that correspond only to positive-signed graphs.

Example. Corresponding to the positive-signed graph G: , we receive the right-handed trefoil knot L: . It is easy to check, by definitions, that $V(\quad) = -t^4 + t^3 + t$ and that $T(\quad) = x^2 + x + y$. Further note that the number of vertices in G is 3, number of vertices in the dual of G is 2, and with of L is 3. Now notice that

$$V_L(t) = (-1)^3 t^{\frac{2-3+3(3)}{4}} T_G(-t, -t^{-1}) = -t^2(t^2 - t - t^{-1}),$$

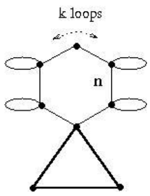
which agrees with the known values.

MAIN RESULTS

In this section we first give the general form of the Tutte polynomial of a family of positive-signed connected graphs, and then specialize it to the Jones, reliability, flow, and chromatic polynomials. Also, we evaluate the Tutte polynomial at some points to get combinatorial information about these graphs.

The Tutte Polynomial

Here we give the general formula of the Tutte polynomial of the following graph. For reference purposes, we denote this graph by $G_{3,.,n,k}$ which has two cycles C_3 and C_n having a common vertex and having k loops. (A loop can be attached to any vertex of the graph.)



Theorem 3.1 [8] If G and G' *are graphs then*

$$T(G \coprod G') = T(G)T(G') \text{ and } T(G * G') = T(G)T(G'),$$

where $G \coprod G'$ *is the disjoint union of* G *and* G' *and* $G * G'$ *is formed by identifying a vertex of* G *and a vertex of* G' *into a single vertex.*

Lemma 3.2. *The Tutte polynomial of the cycle* C_m *of* m *vertices is*

$$T_{C_m}(x, y) = \sum_{i=1}^{m-1} x^i + y. \tag{3.1}$$

Proof. We prove it by induction on m. The Tutte polynomial of the cycle C_3 is $T(\quad) = x^2 + x + y$. To convince ourselves we also give Tutte polynomials of C_4 and C_5 which are:

$T(\quad) = T(\quad) + T(\quad) = x^3 + (x^2 + x + y)$ and $T(\quad) = T(\quad) + T(\quad) = x^4 +$

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$x^3 + x^2 + x + y$.

Suppose the result in (3.1) holds for $m = k - 1$, that is $T_{C_m}(x, y) = \sum_{i=1}^{m-1} x^i + y$.

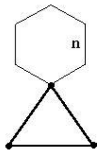
Now taking a cycle C_k with k-vertices, we have $T(C_k) = T(C_k - e) + T(C_k - 1)$. Since $C_k - e$ is a tree with $k - 1$ edges, we have $T(C_k - e) = x^{k-1}$. Now using the inductive step, we get $T(C_k) = x^{k-1} + \sum_{i=1}^{k-2} x^i + y$, which (on collecting the first two terms) finally reduces to the desired results. ■

Proposition 3.3. *The Tutte polynomial of the graph $G_{3,..,n,k}$ is*

$$T_{G_{3,..,n,k}}(x, y) = (x^2 + x + y)(\sum_{i=1}^{n-1} x^i + y)y^k.$$

Proof. The result follows directly from Theorem 3.1 and Lemma 3.2; each loop as a graph contributes a factor of y. ■

If all the k loops are detached from $G_{3,..,n,k}$ we receive the graph on the right, and denote it by $G_{3,..,n}$.



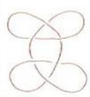
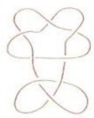
Corollary 3.4. *The Tutte Polynomial of the graph $G_{3,..,n,k}$ is*

$$T_{G_{3,..,n}}(x, y) = (x^2 + x + y)(\sum_{i=1}^{n-1} x^i + y).$$

The Jones Polynomial

The alternating links L that correspond to the graph $G_{3,..,n}$ fall into two categories, the 1-component links (when n is odd) and 2-component links (where n is even).

The graphs along with the corresponding 1-component links (or simply knots) are given in the following table.

n	3	5	7	9	...
G					...
L					...
$a(L)$	5	7	9	11	...
$b(L)$	3	3	3	3	...
$wr(L)$	6	8	10	12	...

Remark 3.5. Please observe from the above table that $a(L) = n + 2$ and $wr(L) = n + 3$.

Proposition 3.6. *The Jones polynomial of the alternating link L that corresponds to the planar graph $G_{3,..,n}$, when n is odd, is*

$$V_L(t) = \left(\frac{t^3 - t^2 - 1}{t + 1}\right)\left(t^{\frac{3n+3}{2}} - t^{\frac{n+5}{2}} - t^{\frac{n+3}{2}} - t^{\frac{n+1}{2}}\right).$$

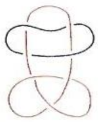
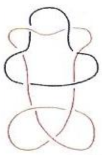
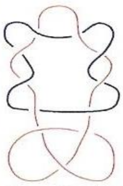
Proof. We prove it by specializing the Tutte polynomial of the graph $G_{3,..,n}$ using Theorem 2.6. Using Remark 3.5, we find that the factor $(-1)^{wr(L)} t^{\frac{b(L)-a(L)-3wr(L)}{4}}$ reduces to $t^{\frac{n+5}{2}}$.

Now

$$
\begin{aligned}
V_L(t) &= t^{\frac{n+5}{2}} T_{G_{3,,n}}(-t,-t^{-1}) \\
&= t^{\frac{n+5}{2}}(t^2-t-t^{-1})[\sum_{i=1}^{n-1}(-t)^i - t^{-1}] \\
&= t^{\frac{n+5}{2}}(t^2-t-t^{-1})[\frac{t^n-t}{t+1}-\frac{1}{t}] \\
&= t^{\frac{n+5}{2}}(\frac{t^3-t^2-1}{t})[\frac{t^{n+1}-t^2-t-1}{t(t+1)}] \\
&= \left(\frac{t^3-t^2-1}{t+1}\right)\left(t^{\frac{3n+3}{2}} - t^{\frac{n+5}{2}} - t^{\frac{n+3}{2}} - t^{\frac{n+1}{2}}\right),
\end{aligned}
$$

as desired. ■

Now, the 2-component links along with the graph $G_{3,,n}$, for even n, can be seen in the following table.

n	4	6	8	10	$\cdots$
G					$\cdots$
L					$\cdots$
$a(L)$	6	8	10	12	$\cdots$
$b(L)$	3	3	3	3	$\cdots$
$wr(L)$	7	9	11	13	$\cdots$

If both the components of the link are oriented either clockwise or counterclockwise direction, then we receive the result:

Proposition 3.7. *The Jones polynomial of the alternating link L that corresponds to the planar graph $G_{3,,n}$, when n is even, is*

$$
V_L(t) = \left(\frac{t^3-t^2-1}{t+1}\right)\left(t^{\frac{3n+3}{2}} + t^{\frac{n+5}{2}} + t^{\frac{n+3}{2}} + t^{\frac{n+1}{2}}\right).
$$

Proof. Similar to the proof of Proposition 3.6; the only difference is that the factor $(-1)^{wr(L)} t^{\frac{b(L)-a(L)-3wr(L)}{4}}$ now reduces to $-t^{\frac{n+5}{2}}$. ■

If one of the components is oriented clockwise and the other counterclockwise, the we get:

Proposition 3.8. *Let L be the link that corresponds to the planar graph $G_{3,,n}$ such that one*

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of the components is oriented clockwise and the other counterclockwise, then

$$V_L(t) = \left(\frac{t^3 - t^2 - 1}{t+1}\right)\left(t^{\frac{-7}{2}} + t^{\frac{-2n-5}{2}} + t^{\frac{-2n-3}{2}} + t^{\frac{-2n-9}{2}}\right).$$

Proof. Similar to the proof of Proposition 3.6; in this case $wr(L) = 3 - n$ and thee factor $(-1)^{wr(L)} t^{\frac{b(L)-a(L)-3wr(L)}{4}}$ reduces to $-t^{\frac{-2n-5}{2}}$. ■

The Flow Polynomial

The flow polynomial was investigated by W. T. Tutte in 1947 in [18] as a function which could count the number of flows in a connected graph.

Definition 3.9. Let G be a graph with an arbitrary but fixed orientation, and let K be an additive abelian group of order $|K|$. A K-flow is a mapping $\emptyset$ of the oriented edges $\vec{E}$(G) into the elements of the group K such that:

$$\textstyle\sum_{\vec{e}=u\to v} \phi(\vec{e}) + \sum_{\vec{e}=u\leftarrow v} \phi(\vec{e}) = 0, \qquad (3.2)$$

for every vertex *v*, and where the first sum is taken over all arcs towards *v* and the second sum is overall arcs leaving *v*.

A K-flow is nowhere zero if $\emptyset$ never takes the value 0. The equation (3.2) is called the conservation law (that is, the Kirchhoff's law is satisfied at each vertex of G).

It is well known [2, 3, 6] that the number of proper K-flow does not depend on the structure of the group, but rather only at its order, and this number is a polynomial function of $|K|$ that we refer to as the *flow polynomial*.

The following, due to Tutte [19], relates the Tutte polynomial of G with the number of nowhere zero flows of G over a finite Abelian group (which, in our case, is Z_k).

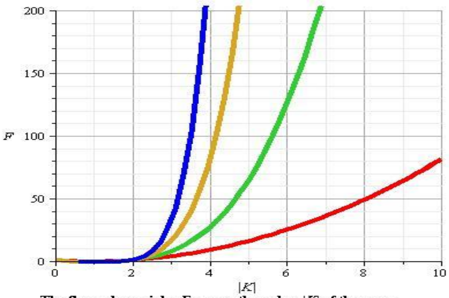
Theorem 3.10. [19] *Let G be a graph and K a finite abelian group. If $F_G(|K|)$ denotes the number of nowhere zero K-flows then*

$F_G(|K|) = (-1)^{|E|-|V|+K(G)} T(0,1-|K|)$.

Proposition 3.11. *The flow polynomial of the graph $G_{3,.,n,k}$ is*

$$F_{G_{3,.n,k}}(|K|) = (|K|-1)^{k+2}.$$

Proof. We prove it by specializing the Tutte polynomial to the flow polynomial by the relation of Theorem 3.10.



The flow polynomials F verses the order $|K|$ of the group
(The curves for n = 0, 1, 2, and 3 appear from right to left.)

Since in the graph $G_{3,.,n,k}$, $k(G) = 1$, $|E| = n + k + 3$, and $|V| = n + 2$, the factor $(-1)^{|E|-|V|+K(G)}$ reduces to $(-1)^{k+2}$. Also, the factor $T(0,1-|K|)$ reduces to $(1-|K|)^{k+2}$. Thus, the flow polynomial of $G_{3,.,n,k}$ finally becomes $(|K|-1)k$+2, as required. ■

The Reliability Polynomial

Definition 3.12. *Let G be a connected graph or network with $|V|$ vertices and $|E|$ edges, and suppose that each edge is independently chosen to be active with probability p. Then the (all terminal) reliability polynomial is*

$R_G(p) = \sum_A p|A| (1-p)^{|E-A|} = \sum_{i=0}^{|E|-|V|+1} g_i\, p^{i-|v|-1}(1-p)^{|E|-i-|v|+1}$,

ISSN: 2186-8476, ISSN: 2186-8468 Print
Vol. 2 No. 2, June 2013

where A is the connected spanning subgraph of G and is the number of spanning connected subgraphs with $i+|V|-1$ *edges.*

Thus, the reliability polynomial, $R_G(p)$, is the probability that there is a path of active edges between each pair of vertices of G.

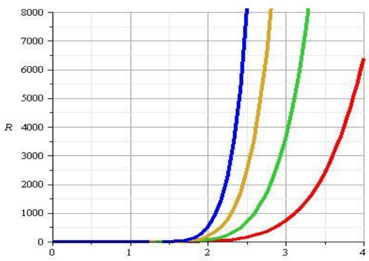
Theorem 3.13. [8] *If G is a connected graph with* $|E|$ *edges and* $|V|$ *vertices, then*

$$R_G(p) = p^{|V|-1}(1-p)^{|E|-|V|+1}T_G(1,\tfrac{1}{1-p}).$$

Proposition 3.14. *The reliability polynomial of* $G_{3,.,n,k}$ is $R_G(p) = p^{n+1}(3\text{-}2p)[n\text{-}(n\text{-}1)p]$.

Proof. We prove it by specializing the Tutte polynomial $T_{G_{3,.,n,k}}$ into the reliability polynomial by using Theorem 3.13.

Since in our case $|V| = n+2$ and $|E| = n+k+3$, the factor $p^{|V|-1}(1-p)^{|E|-|V|+1}$ reduces to $p^{n+1}(1-p)^{k+2}$. Now, the relation



The reliability polynomials *R* verses the probability *p* of an edge
(The curves for n = 3, 4, 5, and 6 appear from right to left.)

$$\begin{aligned} R_G(p) &= p^{n+1}(1-p)^{k+2}\, T_G(1,\tfrac{1}{1-p}) \\ &= p^{n+1}(1-p)^{k+2}\left[\left(1^2+1+\frac{1}{1-p}\right)\left(\sum_{i=1}^{n-1}1^i+\frac{1}{1-p}\right)\left(\frac{1}{1-p}\right)^k\right] \\ &= p^{n+1}(1-p)^{k+2}\left[\left(2+\frac{1}{1-p}\right)\left(n-1+\frac{1}{1-p}\right)\left(\frac{1}{1-p}\right)^k\right] \\ &= p^{n+1}(1-p)^{k+2}\left[\left(\frac{3-2p}{1-p}\right)\left(\frac{n-(n-1)p}{1-p}\right)\left(\frac{1}{1-p}\right)^k\right] \end{aligned}$$

finally reduces to the desired result. ■

Remark 3.15. Since the reliability polynomial is loop independent, the reliability polynomial of $G_{3,.n}$ is the same as the reliability polynomial of $G_{3,.n,k}$.

The Chromatic Polynomial

The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [5] provides an extensive bibliography on the chromatic polynomial, and Dong Koh, and Teo [7] give a comprehensive treatment.

For positive integer λ, a λ-coloring of a graph G is a mapping of $V(G)$ into the set $\{1,2,3,\ldots,\lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices. If $\emptyset$ is a λ-coloring such that $\emptyset(u) \neq \emptyset(v)$ for all $uv \in E$, then $\emptyset$ is called a *proper* (or *admissible*) coloring.

Definition 3.16. The *chromatic polynomial* $P_G(\lambda)$ of a graph G is a one-variable graph invariant and is defined recursively by the following deletion-contraction relation:

$$P_G(\lambda)\, P_G(\lambda) = P(G-e) - P(G/e)$$

In order to find the number of proper λ -colorings of the graph $G_{3,..,n}$, we find the chromatic polynomial of this graph as a special case of the Tutte polynomial $T_{G_{3,..,n}}(x,y)$. The following is the precise relation between these polynomials.

Theorem 3.17. [2] *The chromatic polynomial of a graph G is*

$$P_{G_{3,.,n}}(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_G(1-\lambda,0),$$

where $k(G)$ denote the number of connected components of G.

Proposition 3.18. *The chromatic polynomial of the graph $G_{3,.,n}$ is* $P_{G_{3,.,n}}(\lambda) = (-1)^n(\lambda - 2)(\lambda-1)^2[(1-\lambda)^n - 1]$.

Proof. We prove it by specializing the Tutte polynomial to the chromatic polynomial with the relation

$P_{G_{3,.,n}}(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_{G_{3,.,n}}(1-\lambda, 0)$, which is given in Theorem 3.17.

Since for the graph $G_{3,.,n}$, $|V| = n+2$, and $k(G) = 1$, the factor $(-1)^{|V|-k(G)}\lambda^{k(G)}$ reduces to $(-1)^{n+1}\lambda$.

Now, we have

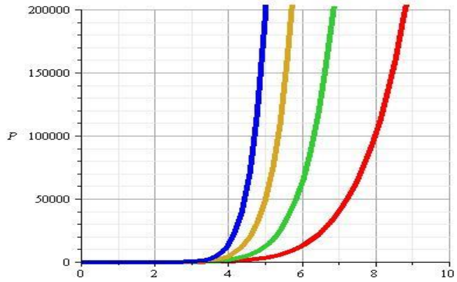
$$P_{G_{3,.,n}}(\lambda) = (-1)^{n+1}\lambda[(1-\lambda)^2 + (1+\lambda)]\left[\sum_{i=0}^{n}(1-\lambda)^i\right]$$

$$= (-1)^{n+1}\lambda[(\lambda-2)(\lambda-1)]\,[\textstyle\sum_{i=0}^{n}(1-\lambda)^i]$$

$$= (-1)^{n+2}\lambda(\lambda-2)[\sum_{i=0}^{n}(1-\lambda)^{i+1}]$$

$$= (-1)^n\lambda(\lambda-2)[(1-\lambda)(\frac{(1-\lambda)^n - 1}{(1-\lambda)-1})]$$

$$= (-1)^n(\lambda-2)(1-\lambda)^2[(1-\lambda)^n - 1],$$



The chromatic polynomials verses number of colors
(The curves for n = 3, 4, 5, and 6 appear from right to left.)

which is the required result. ■

Remark 3.19. Observe that the chromatic polynomial of the graph $G_{3,.,n,k}$ becomes 0; the reason is the common factor y^k.

Subgraphs

The following theorem gives information about the number of different types of subgraphs of a connected graph G.

Theorem 3.20. [8] *If G is a connected graph then:*

1. *$T_G(1,1)$ is the number of spanning trees.*
2. *$T_G(2,1)$ is the number of spanning forests.*
3. *$T_G(1,2)$ is the number of spanning connected subgraphs.*
4. *$T_G(2,2)$ equals $2^{|E|}$, and is the number of subgraphs.*

Proposition 3.21. *The following statements hold for the graph $G_{3,.,n,k}$.*

1. *The number of spanning trees is $3n$.*
2. *The number of spanning forests is $14(2^{n-1} - 1)$.*
3. *The number of spanning connected subgraphs is $2^{n+2}(n+1)$.*
4. *The number of subgraphs is 2^{n+k+3}.*

Proof. We prove it using directly Proposition 3.3 and Theorem 3.20, and using the sums $\sum_{i=1}^{n-1} 1^i = n-1$ and $\sum_{i=1}^{n-1} 2^i 2(2^{n-1}-1)$.

Substituting $x = y = 1$ in Proposition 3.3, we have $T(1,1) = (1+1+1)(\sum_{i=1}^{n-1} 1^i + 1)(1^k) = 3(n-1+1) = 3n$, which is the required result. The proofs of Items 2 and 3 are similar. The proof of Item 4 is straightforward, since $|E| = n+k+3$. ■

Orientations and Score Vectors

The combinatorial interpretations of the Tutte polynomial in Theorem 3.20 are given in terms of the number of certain subgraphs of the graph G. However, they can also be given in terms of orientation of the graph and its score vectors. An orientation of a graph G is the graph $\vec{G}$ all of whose edges are directed.

The score vector of an orientation $\vec{G}$ is the vector $(e_1, e_2, \ldots, e_n)$ such that vertex i has outdegree s_i in the orientation. In the following theorem we gather several similar results about the Tutte polynomial and orientations of a graph.

Theorem 3.22. [8] *if G is a connected graph, then*

1. $T_G(2,0)$ *equals the number of acyclic orientations of G, that is orientations without oriented cycles* [4].
2. $T_G(1,0)$ *equals the number of acyclic orientations with exactly one predefined source v* [22].
3. $T_G(0,2)$ *equals the number of totally cyclic orientations of G, that is orientations in which every arc is a directed cyclic* [22].
4. $T_G(2,1)$ *equals the number of score vectors of orientations of G* [4].

Proposition 3.23. *The following statements hold for the graph $G_{3,..,n,k}$.*

1. *The number of acyclic orientations is* 0.
2. *The number of acyclic orientations with exactly one predefined score v is* 0.
3. *The number of totally cyclic orientations is* 2^{k+2}.
4. *The number of score vectors of orientations is* $7(2^{n-1}+1)$.

Proof. The proofs of Items 1 and 2 are straightforward because when we substitute $y = 0$ in $T_{G_{3,..,n,k}}(x,y) = (x^2+x+y)(\sum_{i=1}^{n-1} x^i + y)y^k$, it becomes 0 due to the factor y^k. For Item 3, we have $T(0,2) = (0^2+0+2)(\sum_{i=1}^{n-1}(0^i+2))(2^k) = 2(0+2)(2^k) = (2^{k+2})$. Similarly, Item 4 can be proved. ■

Proposition 3.24. *The following statements hold for the graph $G_{3,..,n}$.*

1. *The number of acyclic orientations is 3.2^n.*
2. *The number of totally acyclic orientations is 4.*
3. *The number of acyclic orientations with exactly one predefined source v is $2n-2$.*
4. *The number of score vectors of orientations is $7(2^{n-1}+1)$.*

Proof. It is similar to the proof of Proposition 3.23. ■

ACKNOWLEDGEMENT

We are thankful to Higher Education Commission of Pakistan on partially supporting this work.

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